$2\frac{1}{2}$ Hours] [Total Marks: 75

N.B.: (1) **All questions are compulsory.**
(2) Figures to the right indicate marks for respective subquestions.

1. (a) Answer any **ONE**
   i. State and prove the first isomorphism theorem for groups. (8)
   ii. State and prove Cayley's theorem for a finite group. (8)

   (b) Answer any **TWO**
   i. Let G_1, G_2 be groups and H_1, H_2 be normal subgroups of G_1, G_2 respectively. Show that (6)
      p. $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$.
      q. $G_1 \times G_2/H_1 \times H_2$ is isomorphic to $G_1/H_1 \times G_2/H_2$.
   ii. Classify all groups of order ≤ 5 upto isomorphism. (6)
   iii. Find the order of each element of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Is $\mathbb{Z}_2 \times \mathbb{Z}_4$ cyclic? Justify. (6)
   iv. If H is a unique subgroup of order 2 in group G then prove that H is normal in G and $H \subseteq Z(G)$. (6)

2. (a) Answer any **ONE**
   i. Show that a finite integral domain is a field. Give example of an infinite integral domain which is not a field. Is every field an integral domain? (8)
   ii. Let R, R' be commutative rings and $f : R \to R'$ be a ring homomorphism. Prove that: (8)
      p. If I is an ideal of R and f is onto then $f(I)$ is an ideal of R'.
      q. If J is an ideal of R' then $f^{-1}(J)$ is an ideal of R.

   (b) Answer any **TWO**
   i. Define characteristic of ring. Show that char $R = n$ if and only if $o(1_R) = n$ in $(R, +)$. (6)
   ii. Show that there is only one ring homomorphism from $\mathbb{Z}$ into any ring R. (6)
   iii. If I is a proper ideal of $\mathbb{Z}[i]$ then prove that $\mathbb{Z}[i]/I$ is a finite ring. (6)
   iv. If R is an integral domain then, prove that $R[x]$ is also an integral domain. In case R is a field what are the units in $R[x]$? Justify. (6)

3. (a) Answer any **ONE**
   i. Define Euclidean domain.
      Show that the ring of Gaussian integers, $\mathbb{Z}[i]$ is an Euclidean domain. (8)
   ii. Define maximal ideals in a ring R. Show that only maximal ideals in $\mathbb{R}[x]$ are $(x-a)$, $a \in \mathbb{R}$ or (x^2+bx+c), $b, c \in \mathbb{R}$ such that $b^2-4c<0$. Also determine all maximal ideals of $\mathbb{C}[x]$. (8)

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(b) Answer any TWO

i. Show that an ideal P in a commutative ring R is a prime ideal if and only if R/P is an integral domain. (6)

ii. Define gcd of elements of an integral domain. Show that any pair of elements a, b, not both zero, of a principal ideal domain R has a gcd $d \in R$ and $d = ax + by$ for some $x, y \in R$. (6)

iii. Determine which of the following polynomials are irreducible in the indicated rings. (6)
(p) $x^2+2x+1 \in \mathbb{Z}_3[x]$ (q) $2x^3+2 \in \mathbb{Z}_5[x]$ (r) $x^4+10x^2+1 \in \mathbb{Z}[x]$

iv. Let $\phi : R \to S$ be an onto homomorphism. Show that if J is maximal in S then $\phi^{-1}(J)$ is a maximal ideal in R. (6)

4. Answer any THREE

(a) Show that for $G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} / a, b \in \mathbb{R}, a \neq 0 \right\}$, $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} / b \in \mathbb{R} \right\}$ $G/H \cong \mathbb{R}^*$ where $\mathbb{R}^* = \mathbb{R} - 0$ is a group under multiplication (5)

(b) Define normal subgroup of group G. Let $K = \{I, (123), (132)\}$ and $H = \{I, (12)\}$. Is K a normal subgroup of S_3? Is H a normal subgroup of S_3? Justify. (5)

(c) Let R be a ring and $T \subset R, T \neq \phi$. Show that $Z(T) = \{r \in R / rt = tr \text{ for all } t \in T\}$ is a subring of R. (5)

(d) Let $R \neq 0$ be a ring such that $r^2 = r$ for all $r \in R$. Show that R is a commutative ring and has characteristic 2. (5)

(e) Show that $\mathbb{Z}[x]$ has infinitely many maximals ideals. (5)

(f) Define unique factorization domain. Show that $\mathbb{Z}[\sqrt{5}]$ is not a unique factorization domain with the help of $4 = (3 + \sqrt{5})(3 - \sqrt{5})$. (5)

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