15/11/2022

Duration: 2 1/2 Hours **[Total Marks: 75]**

N.B.: **1. All questions are compulsory.**
2. Figures to the right indicate full marks.

Q.1 a) Attempt any ONE question from the following: **(08)**

i. Let G be an abelian group and let a, b ∈ G be such that O (a) = m and O (b) = n. Prove that O (ab) = O (a) O (b) if m and n are relatively prime.

ii. Define group and prove that the set of all integers Z is a group with binary operation defined as a * b = a + b -7.

b) Attempt any TWO questions from the following: **(12)**

i. Let G be a group and H be a subset of G. Prove that H is a subgroup of G if and only if ab^{-1} ∈ H, for all a, b ∈ H.

ii. Let G be a group. Let $a \in G$.Prove that $f: G \to G$ defined by $f(x) = axa^{-1}$ is an isomorphism.

iii. Define Order of an element of a group G. Prove that Order of an element "ab" in a group G is equal to Order of " ba".

Q.2 a) Attempt any ONE question from the following: **(08)**

i. Let $(G_1,*), (G_2,*')$ be groups and let $G_1 \times G_2 = \{(g_1, g_2): g_1 \in G_1; g_2 \in G_2\}$ be the external direct product. Let $a \in G_1, b \in G_2$ be such that O(a) = m and O(b) = n then prove that O((a, b)) = LCM (O(a), O(b)).

ii. State and prove third isomorphism theorem.

b) Attempt any TWO questions from the following: **(12)**

i. Let G, G' be groups and let $f: G \to G'$ be an onto homomorphism. If H' is a normal subgroup of G' then prove that $f^{-1}(H')$ is a normal subgroup of G.

ii. Let G be a group and H be a subgroup of G. Prove that H is a normal subgroup of G if and only if $xHyH = xyH \ \forall \ x, y \in G$.

iii. Let G be a group and let H be a normal subgroup of G. Prove that order of Ha divides the order of a.

Q.3 a) Attempt any ONE question from the following: **(08)**

i. Let G be a cyclic group generated by an element a, then prove that order of a equals the order of the group G.

ii. Prove that a subgroup of a cyclic group is cyclic.

b) Attempt any TWO questions from the following: **(12)**

i. Prove that every finite cyclic group of order n is isomorphic $\mathbb{Z}_n$.

ii. Determine the generators of U (12).

iii. If g is a cyclic group such that $G = < a >$ then prove that $G = < a^{-1} >$.

Q.4 Attempt any THREE questions from the following: **(15)**

a) Let (G, *) be a group. Prove that $a^n * a^m = a^{n+m}$ for $a \in G$ and $n, m \in \mathbb{Z}$

b) Find the order of each element of $(Z_4, +)$.

c) Prove that the set H = {1, -1, I, -i} is a normal subgroup of Q_8.

d) Prove that Z(G) is a normal subgroup of a group G.

e) Determine whether U (20) is a cyclic group.

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