Duration 2 ½Hrs USMT503/02/2022 Marks: 75

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

1. (a) Attempt any One from the following: (8)
   (i) Show that in a metric space (X, d)
      (I) an arbitrary union of open sets is an open set.
      (II) a finite intersection of open sets is an open set.
   (ii) Let (X, d) be a metric space and $A, B \subseteq X$. Show that
      (I) $A \subseteq B \implies A^\circ \subseteq B^\circ$
      (II) $(A \cap B)^\circ = A^\circ \cap B^\circ$
      (III) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ and the inequality may be strict.

   (b) Attempt any Two from the following: (12)
   (i) Let (X, d) be a metric space and $d_1 : X \times X \to \mathbb{R}$ be a metric defined as $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$, $\forall x, y \in X$. Show that d and d_1 are equivalent metrics on X.
   (ii) Prove that $\overline{A}$ is a closed subset in a metric space (X, d).
   (iii) Prove that $(l_1, \|.\|_1)$ is a normed linear space.

2. (a) Attempt any One from the following: (8)
   (i) Show that $[0, 1]$ is uncountable.
   (ii) Let (X, d) be a metric space and A be a subset of X. Show that $p \in X$ is a limit point of A if and only if there is a sequence of distinct points in A converging to p.

   (b) Attempt any Two from the following: (12)
   (i) Prove that a subspace (Y, d) of a complete metric space (X, d) is complete if Y is closed. Can we say $(1, 2)$ is complete in $(\mathbb{R}, d_1)$, where d_1 is discrete metric? Justify.
   (ii) Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = (x - a)^2(x - b)^2 + x$ takes the value $\dfrac{a + b}{2}$ for some value of $x \in \mathbb{R}$. (distance in $\mathbb{R}$ being usual.)
   (iii) Let (X, d) be a metric space. If (x_n) and (y_n) are sequences in X such that $x_n \longrightarrow p$ and $y_n \longrightarrow q$ then prove that $d(x_n, y_n) \longrightarrow d(p, q)$ in $\mathbb{R}$ with usual distance.

3. (a) Attempt any One from the following: (8)
   (i) Show that a compact subset of a metric space is closed and bounded.
   (ii) Let $K \subseteq \mathbb{R}^n$ with Euclidean metric. Prove that K is sequentially compact iff K satisfies Bolzano Weierstrass Property.

   (b) Attempt any Two from the following: (12)
   (i) Show that $X = (0, 1)$ under usual metric is not compact.
   (ii) Check whether following subset of $\mathbb{R}^2$ is compact under Euclidean metric: $K = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$

(iii) Prove that an arbitrary intersection of compact sets is compact. Justify whether an arbitrary union of compact sets is compact?

4. Attempt any Three from the following: (15)

(a) Show that $G = \{(x, y) \in \mathbb{R}^2 : x > 0, y < 0\}$ is an open subset of $\mathbb{R}^2$ with respect to the Euclidean metric.

(b) Define distance of a point p from a set A. Find the distance of point $p = 0$ from the following sets in $\mathbb{R}$ with d: usual distance:

i. $A = [1, 2]$
ii. $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$
iii. $A = \{10, 20, 30\}$

(c) Let d_1 and d_2 be metrics on a non-empty set X such that there exist $k_1, k_2 > 0$ such that $k_1 d_1(x, y) \leq d_2(x, y) \leq k_2 d_1(x, y)$, $\forall x, y \in X$. Show that a sequence (x_n) is Cauchy in (X, d_1) if and only if sequence (x_n) is Cauchy in (X, d_2).

(d) Prove that a discrete metric space is a complete metric space.

(e) Prove or disprove: Interior and closure of a compact set are compact sets.

(f) Prove that $K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is a compact subset of $\mathbb{R}$ with usual distance.

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