T.Y.B.Sc Sem VI
Com April 2018
Math 8

Q.P. Code: 33539

Time: 2:30 Hours | Total Marks: 75

N.B: 1) All questions are compulsory
2) From questions 1, 2 and 3 attempt any one from part (a) and any two from part (b)
3) Attempt any three From question 4.
4) Figures to the right indicate marks

Q.1 (a) i) If $\{f_n\}$ and $\{g_n\}$ are sequences of real valued functions converging uniformly to f and g respectively on [a, b] then prove or disprove 8
1. $\{f_n + g_n\}$ converge uniformly to f + g.
2. $\{f_n g_n\}$ converge uniformly to fg.

ii) Let $\{f_n\}$ be a sequence of real valued functions continuous on a non- empty subset S of $\mathbb{R}$. If $\sum_{n=1}^{\infty} f_n$ converges uniformly to a function f on S, then show that f is continuous on S. Further show that for each $p \in S$, $\lim_{x\to p} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x\to p} f_n(x)$ 8

(b) i) Discuss the pointwise and uniform convergence of the series of functions $\sum_{n=1}^{\infty} \frac{x^4}{(1+x^4)^n}$ on $[0, \infty)$ 6

ii) Let $f_n: [0,1] \to \mathbb{R}$ be defined as $f_n(x) = \begin{cases} 0 & for \quad 0 \le x < \frac{1}{n} \\ n & for \quad \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & for \quad x > \frac{2}{n} \end{cases}$ 6

Check whether $\int_0^1 \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx$. Does $\{f_n\} \to f$ uniformly on [0, 1]? Justify.

iii) If a real power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence r, then show that it converges uniformly on [-s, s] where $0 \le s < r$. 6

iv) Let $f_n: [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx}{1+nx}$ for $x \in [0,1]$. Find the pointwise limit of $\{f_n\}$ on [0, 1]. Does $\{f_n\}$ converge uniformly on [0, 1]? Justify. 6

Q.2 (a) i) When is a function $f: A \subseteq \mathbb{C} \to \mathbb{C}$ said to be differentiable at $z_0 \in A$? Show that if f is differentiable at z_0 then it is continuous at z_0. If $f'(z_0)$ and $g'(f(z_0))$ exists then prove that composite function $F(z) = (gof)(z)$ is differentiable at z_0 and $F'(z_0) = (gof)'(z_0) = g'(f(z_0)).f'(z_0)$. 8

ii) Let $\Omega \subset \mathbb{C}$ be a domain in $\mathbb{C}$. If $u, v: \Omega \to \mathbb{R}$ are such that u_x, v_x, u_y, v_y exist, satisfy Cauchy- Riemann equations and u_x, v_x, u_y, v_y are continuous on Ω, prove that $f(z) = u(x, y) + iv(x, y)$ is analytic in Ω. 8

(b) i) Using the definition of differentiability, discuss the differentiability of the function $f(z) = |z|^2$ for any $z \in \mathbb{C}$. Is f analytic on $\mathbb{C}$? Justify. 6

ii) If z_0 and w_0 are points in the z and w planes respectively, then prove that $\lim_{z\to\infty} f(z) = w_0$ if and only if $\lim_{z\to 0} f(\frac{1}{z}) = w_0$. Hence show that $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4$ 6

iii) Let D be a domain in $\mathbb{C}$. $f(z)$ and $\overline{f(z)}$ are analytic in D. Show that $f(z)$ must be constant throughout D. 6

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iv) Show that $u(x,y) = xy^3 - x^3y$ is a harmonic function and find its harmonic conjugate $v(x,y)$. 6

Q.3 (a) i) f is analytic inside and on a simple closed curve C, taken in the positive sense. Prove that $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}$ where z is any point inside C. 8

ii) Let C be a simple closed curve in the interior of the disc of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. Prove that in the interior of the disk of convergence $S'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$. 8

(b) i) State and prove Cauchy Goursat theorem. (Weaker form). 6

ii) State Laurent's Theorem. Expand $f(z) = \frac{1}{z(z-1)}$ as a Laurent series in the domain $1 < |z-2| < 2$. 6

iii) State Cauchy Residue theorem. Compute the residue of $f(z) = \frac{z^2}{(z-1)^2(z-2)(z-3)}$ at its double pole. 6

iv) Show that $f(z) = \frac{1}{1-z}$ has its Taylor series representation about the point $z_0 = i$ given by $f(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$ for $|z-i| < \sqrt{2}$. 6

q.4 i) By integrating a suitable power series term by term, show that $\tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $|x| < 1$. 5

ii) Discuss the pointwise and uniform convergence of the sequence of functions $\{x^2e^{-nx}\}$ on $[0, \infty)$. 5

iii) Find the image of the set $\{z: \mathrm{Re}(z) \geq \frac{1}{2}\}$ under the reciprocal map $w = \frac{1}{z}$ in the extended complex plane. 5

iv) Find the bilinear transformation $w = S(z)$ that maps the points $z_1 = 1, z_2 = i, and\ z_3 = -1\ onto\ w_1 = 0, w_2 = 1\ and\ w_3 = \infty$ respectively. 5

v) Evaluate $\int_C \frac{dz}{(z^2+1)(z^2-4)}$ where C is the circle $|z| = 1.5$ by using Cauchy integral formula. 5

vi) State and prove ML inequality and hence show that $\left|\int_C \frac{dz}{z^2-1}\right| \leq \frac{\pi}{3}$ where C is the part of $|z| = 2$ in the first quadrant. 5

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