To pubs

Con. 2403-13. **CG-2323**

(2½ Hours) **[Total Marks : 60**

N.B. : (1) **All** questions are **compulsory**.
(2) **Figures** to the **right** indicate **marks** for respective **subquestions**.

1. (a) Let (X, d) be a metric space and Y be a non-empty subset of X. Show that a subset G of Y is open in the subspace (Y, d) if and only if $G = V \cap Y$ where V is an open set in (X, d). (7)
   (b) Attempt any **two** questions:
      (i) State and prove Hausdorff property in a metric space (X, d). (4)
      (ii) Let (X, d) be a metric space. Show that $\overline{B(x,r)} \subseteq B[x,r]$, where $B(x,r) = \{y \in X : d(x,y) < r\}$, $B[x,r] = \{y \in X : d(x,y) \leq r\}$. Give an example to show that the strict inequality may hold. (4)
      (iii) Let $X = C[0,1]$ and d_1 and d_∞ be the metric on X induced by $\| \ \|_1$ and $\| \ \|_\infty$, where $\|f\|_1 = \int_0^1 |f(t)|dt$ and $\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$. Prove or disprove: d_1 and d_∞ are equivalent metrics on X. (4)
      (iv) Let (X, d) be a metric space and $A \subseteq X$. Prove that $diam(A) = diam(\overline{A})$. (4)
2. (a) Let (X, d) be a complete metric space. Prove that a subspace (Y, d) of (X, d) is complete if and only if Y is a closed. (7)
   (b) Attempt any **two** questions:
      (i) Let (X, d) be a metric space. Show that a convergent sequence in X is Cauchy. Give an example to show that the converse is not true. Further. show that a Cauchy sequence (x_n) in X is convergent if it has a convergent subsequence. (4)
      (ii) Prove that $(\mathbb{R}^2, d)$ is a complete metric space, where $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$ for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. (4)
      (iii) Check if Cantor's Theorem is applicable in the following examples. Also, find $\cap_{n \in \mathbb{N}} F_n$ in each case, where (F_n) is a sequence of subsets of $\mathbb{R}$ and the distance in $\mathbb{R}$ is usual. (4)
         (i) $F_n = [n, \infty)$ (ii) $F_n = (0, \frac{1}{n})$
      (iv) Prove that $(\mathbb{N}, d_1)$ is complete, where d_1 is the metric defined by (4)

$$d_1(m,n) = \begin{cases} 0 & \text{if } m = n \\ 1 + \frac{1}{m+n} & \text{if } m \neq n \end{cases}$$

3. (a) Let (X, d) and (Y, d') be metric spaces and $f : X \to Y$. Show that the following statements are equivalent. (7)
      (i) f is continuous on X.
      (ii) For each open subset G of Y, $f^{-1}(G)$ is an open subset of X.
   (b) Attempt any **two** questions:
      (i) Let (X, d) and (Y, d') be metric spaces. Show that $f : X \to Y$ is continuous on X if and only if for each subset B of Y, $(\overline{f^{-1}(B)}) \subseteq f^{-1}(\overline{B})$. (4)
      (ii) Let (X, d) and (Y, d') be metric spaces and $f, g : X \to Y$ be continuous maps. Show that $A = \{x \in X : f(x) = g(x)\}$ is a closed subset of X. (4)
      (iii) Let (X, d) and (Y, d') be metric spaces and $f : X \to Y$ be a continuous onto map. If D is a dense subset of X, show that $f(D)$ is a dense subset of Y. (4)
      (iv) If $f : \mathbb{R} \to \mathbb{R}$ is such that $f'(x)$ exists for all $x \in \mathbb{R}$ and $|f'(x)| \leq M$ for all $x \in \mathbb{R}$, then show that f is uniformly continuous on $\mathbb{R}$. (4)
4. Attempt any **three** questions:
   (a) Show that $A = \{x \in \mathbb{Q} : -\sqrt{2} < x < \sqrt{2}\}$ is both open and closed in the subspace $\mathbb{Q}$ of $\mathbb{R}$ with usual distance. (5)

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(b) Let d_1 and d_2 be metrics on a non-empty set X such that there exist $k_1, k_2 > 0$ such that

$$k_1 d_1(x,y) \leq d_2(x,y) \leq k_2 d_1(x,y) \; \forall \; x, y \in X.$$

Then show that (x_n) is Cauchy in (X, d_1) if and only if (x_n) is Cauchy in (X, d_2). (5)

(c) Let (X, d) be a metric space and let $A \subseteq X$. If $d_A : X \to \mathbb{R}$ is defined by $d_A(x) = d(x, A)$, then show that d_A is continuous on X. (distance in $\mathbb{R}$ is usual). (5)

(d) Prove or disprove: Let (X, d) and (Y, d') be metric spaces. If (X, d) is complete and $f : X \to Y$ is continuous and onto, then (Y, d') is complete. (5)