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Topic : Integer Programming Problem (All IPP)

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Integer Programming is a special class of LPP where all or some variables are constrained to be integer. In real life problems many a times it is essential to have only integer solutions. e.g. if x_1 is no. of batches, $x_1 \geq 0$ and integer. One can think that, obtain the sol. of problem & if it is a fractional, then round it off to get an integer value. But this method do not give us correct value because there is no guarantee that deviation from exact integer sol. will not be large enough to have no effect on feasibility condition. In such cases special technique is applied.

All & mixed up IPP

Let the primal be Max. Z = CX, subject to, AX ($\geq = \leq$) B, where $X \geq 0$ s.t. X_j being an integer value for some j = 1,2……k

If i) k = n, i.e. all variables are integers then the problem is called All IPP.

ii) k < n i.e. only some variables are integers, then it is called mixed IPP.

Method 1: Graphical method

If the given IPP has m constraints in two variables, we can use graphical method.

Steps :

Ignoring integrality condition solve given LPP using usual graphical method.

1) If sol. Is integral, procedure ends, but if it is non integral, proceed to next step.
2) In order to obtain all feasible & integer sols. So that we can select one sol. As the best sol.,
3) find all points (x,y) in feasible region s.t. x,y € I. Find value of Z at all these points & select that pt. as optimum sol. which optimizes Z.

If no. of variables is > 2 Graphical method fails. We use general integer Programming Algorithm.

Strategy of these algorithms involves following three phases (steps)

1) Relax the integrality conditions on decision variables which turns IPP into LPP. Solve LPP, & identify its continuous optimum.
2) Starting from continuous optimum pt., add special constraints that iteratively modify the LP
3) continuous sol. space, which eventually renders an optimum extreme pt. satisfying the integer requirements.

Two general methods have been developed for generating the special constraints.

B.&B. method i.e. Branch &Bound Method.(1960)

1) Cutting Plane Method i.e. Gomory's Method.(1956)
2) B.& B. Method is far more successful than the cutting plane method.

B. & B. Method.(general A. Land & G. Doig algorithm)

Let the problem be of maximization. Set an initial lower bound $Z = -\infty$ (if minimization set initial upper bound $Z = \infty$)

Steps :

0) Relax integrality condition & obtain new LPP. Name this problem as LP0. & solve it. Branch this LP0 in two subsets, one subset expected to contain sol. & other not expected to contain opt. sol. Name these subsets as LP1& LP2. In general let LPi & LP_{i+1} be such sub problems.
1) Let LPi be next sub problem to be examined. Solve LPi with additional condition, & attempt to fathom it using one of three conditions.
   a) The optimal Z value of LPi cannot yield a better objective value than the current lower bound.
   b) LPi yields a better feasible integer sol. than current lower bound.
   c) LPi has no feasible sol.

   Two cases will arise,
   1) If LPi is fathomed & a better sol. is found, update the lower bound. Now try for next branch.i.e. other sub problem.(L_{Pi+1})If all sub problems have been fathomed. Stop; the optimum ILP is associated with the current lower bound, if any, o.w. solve LP_{i+1} using step 1
   2) If LP_i is not fathomed go to step 2 for branching.
2) (Branching) Select one of the integer variable x_j, whose optimum value x_j^* in LP_i sol. is not integer. Eliminate the region $[x_j^*] < x_j < [x_j^*] +1$, where $[v]$ is largest integer $\leq v$. & concentrate on region $x_j \leq [x_j^*]$ & $x_j \geq [x_j^*] +1$, creating two sub problems. Repeat Step 1 till we obtain optimum integer sol. of given IPP.

Cutting Plane Method / Gomory's Method.

Given problem is solved with usual Simplex method. If all variables are integers then given opt. sol. is IPP sol. but if some are non integer then given LPP is modified by inserting a new constraint, called Gomory's or secondary constraint which eliminates some non integer variables without affecting integer variable. After adding Gomory's constraint the problem is solved by dual Simplex method to get an opt. integer sol.

Construction of Gomory's constraint :

Consider LPP for which an opt. non integer b.f.sol. has been obtained. With usual notation, let final table be given as

C_B	x_B	B	X_1	X_2	S_1	S_2
C_{B1}	X_2	$\hat{b}_1$	Y_{11}	1	0	Y_{14}
C_{B2}	S_1	$\hat{b}_2$	Y_{21}	0	1	Y_{24}
	Z_j	$Z_{opt.}=y_{00}$	Z_1	Z_2	Z_3	Z_4

Let X_B be non integer sol. Let $\hat{b}_1$ be non integer sol.

$\hat{b}_1$ = $Y_{11}X_1 + X_2 + 0S_1 + Y_{14}S_2$ …………..(1)

$\hat{b}_1 \geq 0$,hence fractional part of $\hat{b}_1$ also must be non negative.

Split each of Y_{1j} (which may be +ve or –ve) in (1) into an integer part I_{1j} & nonnegative fractional part f_{1j}, $o \leq f_{1j} \leq 1$ { e.g. if Y_{11} = 3/2 =1+ ½,
if Y_{14} = -3/2 =-2 + ½ }

$\hat{I}_1 + \hat{f}_1 = (I_{11} + f_{11}) X_1 + X_2 + (I_{14} + f_{14}) S_2$

$\hat{f}_1 + (\hat{I}_1 - I_{11} X_1 - I_{14} S_2 - X_2) = f_{11} X_1 + f_{14} S_2$

$\hat{f}_1$ + integer = $f_{11} X_1 + f_{14} S_2 \geq \hat{f}_1$

$\hat{f}_1 \leq f_{11} X_1 + f_{14} S_2$, $\hat{f}_1 + G_{slack}^{(1)} = f_{11} X_1 + f_{14} S_2$ ……..(2)

Where $G_{slack}^{(1)}$ is first Gomory's slack in Gomory's constraint & X_1, S_2 are non basic variables. Additional constraint (2) is to be included in opt. table. Apply Dual Simplex method to solve such problem.

<u>All IPP Iterative procedure.</u>

<u>Steps :</u>

1) Write new problem with <u>min. of Z</u> (For convenience) & relaxing integrality conditions.
2) Introduce slack & surplus variable & solve problem with usual Simplex method.
3) Test the integrality of opt. sol.
   a) If opt. sol. Includes all integer values, an opt. basic feasible integer sol. Has been obtained.
   b) If opt. sol. does not satisfy integrality condition, go to next step.
4) Examine the constraint equations corresponding to the current opt. sol. Let these equation be represented as

   $$\hat{b}_i = \sum_{j=1}^{n} \hat{Y}_{ij} X_j, \quad i = 1,2 \text{ ……..m} \text{ (n= no. of variables, m= no. of constraints.)}$$

   Choose the largest fraction of $\hat{b}_i$'s. Let it be $\hat{f}_k$

5) Express each of fraction in k^{th} row of the opt. Simplex table as sum of the integer & a non –ve fraction.
6) Find Gomorian constraint as

   $\sum_{j=1}^{n} f_{kj} x_j \geq \hat{f}_k$ & obtain Gomory's constraint introducing Gomory's slack as

   $\hat{f}_k + G_{slck}^{(1)} = \sum_{j=1}^{n} f_{kj} x_j$, Augment it to current set of equations.

7) Starting with this new set of equations, find the new opt. sol. by Dual Simplex method.

8) If this new opt. sol. for modified LPP is an integer sol., it is also feasible & opt. sol. for given IPP, o.w. return to step 4 & repeat the procedure until an opt. feasible integer sol. has been obtained.

N.B. 1) The algorithm requires that all the coefficients in the constraints and all the R.H.S. values of the constraints be integers.

2) At any stage of algorithm, when two or more constraints have fractional values on the R.H.S., it is desirable to generate the cut by using the constraint whose R.H.S. has a fractional part close to ½.