T.Y. B.Sc sem VI Algebra (Rev)

Exam April 2018

Maths Revised

QP Code : 33324

Maths (Rev)

13/04/18

$2\frac{1}{2}$ Hours] [Total Marks: 75

N.B.: (1) All questions are compulsory.
(2) Figures to the right indicate marks for respective subquestions.

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1. (a) Attempt any **ONE**.
   i. State and prove the Cayley's theorem for finite groups. (8)
   ii. If $(G_1, \cdot), (G_2, *)$ are groups and $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ with binary operation $\circ$ defined by $(g_1, g_2) \circ (g_1', g_2') = (g_1 \cdot g_1', g_2 * g_2')$ then show that, if $H_1, H_2$ are normal subgroups of $G_1, G_2$ respectively then $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$ and $\frac{G_1 \times G_2}{H_1 \times H_2}$ is isomorphic to the external direct product $\frac{G_1}{H_1} \times \frac{G_2}{H_2}$. (8)

   (b) Attempt any bf TWO
   i. Let $H$ be a normal subgroup of $G$ and $K$ be any subgroup of $G$, show that $HK = KH$. (6)
   ii. Show that a finite cyclic group of order $n$ is isomorphic to $\mathbb{Z}_n$. (6)
   iii. Which of the following pairs are isomorphic? (6)
   (p) $U(10)$ and $V_4$ (q) $S_3$ and $GL(2, \mathbb{Z}_2)$, the group of $2 \times 2$ invertible matrices over $\mathbb{Z}_2$. Justify.
   iv. Let $H$ be a normal subgroup of order 2 in $G$. Show that $H \subseteq Z(G)$. Further if $G$ is of order 10 show that $G$ is Abelian. (6)

2. (a) Answer any **ONE**
   i. Show that, characteristic of a ring $R$ is $n$ if and only if the order of the multiplicative identity of $R$ is $n$ in the group $(R, +)$. Find characteristic of $\mathbb{Z}_4 \times \mathbb{Z}_6$. (8)
   ii. Let $R$ be a commutative ring. If $I, J$ are ideals in $R$,which of $I \cup J$, $I \cap J$, $I + J$ and $IJ$ are ideals of $R$, where $I+J = \{x+y : x \in I, y \in J\}$ and $IJ = \left\{ \sum_{i=1}^{n} x_i y_i \ : \ x_i \in I, y_i \in J, n \in \mathbb{N} \right\}$? Justify. Further if $R = I + J$, show that $I \cap J = IJ$. (8)

   (b) Answer any **TWO**
   i. State and prove the First Isomorphism Theorem(Fundamental theorem of homomorphism) of rings. (6)
   ii. Show that, $\bar{J}$ is an ideal of the quotient ring $R/I$ if and only if there is an ideal $J \subseteq I$ of the ring $R$ such that $\bar{J} = \{x + I : x \in J\}$. (6)

iii. Let $R$ be a commutative ring. Show that $I = \{a : a \in R, a^n = 0 \text{ for some } n \in \mathbb{N}\}$ is an ideal of $R$ and $R/I$ has no nilpotent element. (6)

iv. Show that $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^2+x+1)$ are non-isomorphic rings. (6)

3. (a) Answer any **ONE**

i. Let $R$ be a commutative ring. Show that an ideal $M$ is maximal in $R$ if and only if $R/M$ is a field. (8)

ii. Let $R$ be an Integral Domain. Let $p \in R$. Then, (8)

(p) $p$ is prime iff $(p)$ is a non zero prime ideal of $R$.

(q) If $p$ is prime then $p$ is irreducible in $R$.

(b) Answer any **TWO**

i. Let $\mathbb{F}$ be a field. Let $f(x) \in \mathbb{F}[x]$ with $\deg f(x) = 2$ or 3. Show that $f(x)$ is reducible over $\mathbb{F}$ if and only if $f(x)$ has a zero in $\mathbb{F}$. (6)

ii. Show that a commutative ring is a field if and only if it has no ideals except 0 and itself. (6)

iii. Show that $\mathbb{Z}[i]/(2+i)$ is isomorphic to $\mathbb{Z}_5$. (6)

iv. Let $p$ be a prime integer. Show that the number of monic quadratic reducible polynomials in $\mathbb{Z}_p[x]$ is $p(p+1)/2$. Determine the number of monic quadratic irreducible polynomial in $\mathbb{Z}_p[x]$. (6)

4. Answer any **THREE**

(a) If a cyclic subgroup $H$ of $G$ is a normal subgroup of $G$, then show that every subgroup of $H$ is normal in $G$. (5)

(b) Let $G = A_4$, $K = \{I, (12)(34), (13)(24), (14)(23)\}$, $H = \{I, (12)(34)\}$. Show that $H$ is a normal subgroup of $K$ but $H$ is not a normal subgroup of $G$. (5)

(c) Show that $S = \{a + ib : a, b \in \mathbb{Z}, b \text{ is even}\}$ is a subring but not an ideal of $\mathbb{Z}[i]$. (5)

(d) Show that $\mathbb{Z}[\sqrt{5}]$ has infinitely many units. (5)

(e) Let $\mathbb{F}$ be a field and '$a$' be a non-zero element in $\mathbb{F}$. If $f(x+a)$ is irreducible in $\mathbb{F}[x]$, prove that $f(x)$ is irreducible in $\mathbb{F}[x]$. Use this result to show that the polynomial $8x^3 - 6x + 1$ is irreducible over $\mathbb{Q}$. (5)

(f) Let $\mathbb{F}$ be a field. Show that $f(x)$ and $g(x)$ are associates in $\mathbb{F}[x]$ if and only if $f(x) = cg(x)$ where $c(\neq 0) \in \mathbb{F}$. (5)

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