V.P.M.'S B. N. BANDODKAR COLLEGE OF SCIENCE, THANE

S.Y.B.Sc. Semester IV End Examination, March 2017

USMT 402

Duration: 2 hours 30 mins **Total Marks:75**

Instructions to the candidates:

1. All questions are compulsory.
2. Figures to the right indicate marks.
3. Calculators are not allowed.

Q.1 A) Attempt any one of the following:

1. Let G be a group and $a, b \in G$ be such that $o(a) = n,\ o(b) = m, ab = ba$ and $(m, n) = 1$. Prove that $o(ab) = o(a)o(b)$. 8
2. Define Group. For any positive integer n, prove that $U(n) = \{\bar{a} \in \mathbb{Z}_n : (a, n) = 1\}$ is a group under multiplication modulo n. 8

B) Attempt any two of the following:

1. Prove that $\mathbb{C}^*$ is a group under multiplication. 6
2. Let G be a group. For any $a, b \in G$, prove that $(aba^{-1})^n = ab^na^{-1},\ \forall n \in \mathbb{Z}$. 6
3. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \in GL_2(\mathbb{R})$. Find $o(A), o(B)$ and $o(AB)$. 6
4. Define subgroup. Prove that $SL_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$. 6

Q.2 A) Attempt any one of the following:

1. Prove that subgroup of a cyclic group is cyclic. 8
2. Find the following for group $U(11)$: 8
   (i) Order of each element (ii) All subgroups (iii) Generators of each subgroup.

B) Attempt any two of the following:

1. Let G be a cyclic group of order n. Prove that G has $\varphi(n)$ generators. 6
2. Prove that $H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$ is a cyclic subgroup of $GL_2(\mathbb{R})$. 6
3. Prove that infinite cyclic group has infinitely many subgroups. 6
4. Find all the generators of $\mathbb{Z}_9$. 6

P.T.O.

Q.3 A) Attempt any one of the following:

1. State and prove Euler's theorem and Fermat's theorem. 8

2. Prove that every infinite cyclic group is isomorphic to $\mathbb{Z}$. 8

B) Attempt any two of the following:

1. Let G and G' be groups and $f: G \to G'$ be a homomorphism. Prove that f is one-one iff $\ker f = \{e\}$. 6

2. Let G be a group and $H \leq G$. Prove that $o(Ha) = o(Hb)$ for any $a, b \in G$. 6

3. Let G and G' be groups and $f: G \to G'$ be an isomorphism. Prove that G is abelian iff G' is abelian. 6

4. Let G be a group and H and K be distinct subgroups of G of prime order p. Prove that $H \cap K = \{e\}$. 6

Q.4 Attempt any three of the following:

1. Let $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \middle| a \in \mathbb{R}, a \neq 0 \right\}$. Show that G is a group under matrix multiplication. 5

2. Let G be a group and $H, K \leq G$. Prove that $H \cap K \leq G$ but $H \cup K$ may not be a subgroup of G. 5

3. Find the unique subgroup of $\mathbb{Z}_{28}$ of order 7. 5

4. Prove that any group of order 3 must be cyclic. 5

5. Let $G = \mathbb{Z}$ and $H = n\mathbb{Z}$ for $n \in \mathbb{N}$. Prove that $[G: H] = n$. 5

6. Let $f: \mathbb{C} \to \mathbb{C}$ be defined as $f(z) = \bar{z}$. Prove that f is an automorphism. 5

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