Equation of motion of a vibrating string:

Consider a string of finite length ‘L’ fixed at the rigid supports. If it is plucked and released, it vibrates. Our aim is to study motion of this vibrating string. For this purpose an equation governing the motion of the string is to be obtained using Newton's laws of motion.

There are infinitely many particles on the string. Hence applying Newton’s laws to these particles, we get infinitely many equations. Instead of this, we employ Euler’s approach.

Consider an element ‘dl’ of the string. Newton’s law can be applied to such an element. We introduce two co-ordinates, x which changes continuously from zero until the end of the string. The other dependant variable is the displacement of the element of the string at position x from the mean position in the vertical direction. Variable u is a function of position and time t i.e. \( u(x, t) \).

Following are the assumptions used in obtaining the equation of motion of the vibrating string.

1) The string vibrates in the vertical plane only (No sidewise swing)
2) Each particle on the string vibrates along the vertical directions only (No sidewise deflection).
3) The Tension in the string \( \tau \) is small and constant throughout the length.

Since the element \( dx \) vibrates along the vertical direction only, the net force must be along the vertical direction. The net force acting on the element \( dx \) along vertical direction can be calculated as follows:

\[
f_{net} = (\tau \sin \theta)_{x + dx} - (\tau \sin \theta)_{x} \\
\Rightarrow \theta \rightarrow small \quad \sin \theta \approx \tan \theta \\
f_{net} = (\tau \tan \theta)_{x + dx} - (\tau \tan \theta)_{x} \\
= \left( \tau \frac{\partial u}{\partial x} \right)_{x + dx} - \left( \tau \frac{\partial u}{\partial x} \right)_{x} \\
f_{net} = \lim_{dx \rightarrow 0} \frac{dx}{dx} \left( \tau \frac{\partial u}{\partial x} \right)_{x + dx} - \left( \tau \frac{\partial u}{\partial x} \right)_{x} dx
\]
\[ f_{net} = \frac{\partial}{\partial x} \left( \tau \frac{\partial u}{\partial x} \right) \]

\[ f_{net} = \tau \frac{\partial^2 u}{\partial x^2} (x,t) \text{ } dx \]

(1)

According to Newton’s 2\textsuperscript{nd} law of motion. This net force must be equal to the product of mass and acceleration of the element in the vertical direction.

\[ : (\text{Mass})(\text{Acceleration}) = f_{net} \]

\[ \sigma \text{ } dx \text{ } \frac{\partial^2 u}{\partial t^2} = \tau \frac{\partial^2 u}{\partial x^2} \text{ } dx \]

\[ \frac{\partial^2 u}{\partial x^2} (x,t) \left( \frac{\sigma}{\tau} \right) \frac{\partial^2 u}{\partial t^2} (x,t) = 0 \]

Where \( \sigma \) - linear mass density of a string i.e. mass per unit length of the string

\[ \frac{\partial^2 u}{\partial x^2} (x,t) - \left( \frac{1}{\tau/\sigma} \right) \frac{\partial^2 u}{\partial t^2} (x,t) = 0 \]

\[ \frac{\partial^2 u}{\partial x^2} (x,t) - \left( \frac{1}{\sqrt{\tau/\sigma}} \right)^2 \frac{\partial^2 u}{\partial t^2} (x,t) = 0 \]

\[ \frac{\partial^2 u}{\partial x^2} (x,t) - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} (x,t) = 0 \]

Where \( c = \frac{\sqrt{\tau/\sigma}}{\sigma} \) is velocity of propagation of waves along the string

This is the required equation of motion of the vibrating string.

Normal Modes of Vibration: -

The 2\textsuperscript{nd} order partial linear different equation with constant coefficient satisfied by the vibrating string (equation of motion of the vibrating string), is given by

\[ \frac{\partial^2 u}{\partial x^2} (x,t) - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} (x,t) = 0 \]

This partial different equation is subject to the following boundary condition as the string is fixed at its two ends to the rigid support.

\[ u(0,t) = 0 \text{ } \text{ or } \text{ } u(x,t) \bigg|_{x=0} = 0 \]

\[ u(L,t) = 0 \text{ } \text{ or } \text{ } u(x,t) \bigg|_{x=L} = 0 \]

In order to solve the above partial differential equation we employ the method of separation of variables. The solution \( u(x,t) \) can be expressed as a product of two functions such that each function is a function of only one independent variable.

Let,

\[ u(x,t) = X(x) \Theta(t) \]

(1)

\[ \frac{\partial^2 u}{\partial x^2} (x,t) = \Theta(t) \frac{d^2 X(x)}{dx^2} \]

and
\[ \frac{\partial^2 u}{\partial t^2}(x,t) = X(x) \frac{d^2 \Theta(t)}{dt^2} \]

Substituting this in equation of motion

\[ \frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \]

\[ \Theta(t) \frac{d^2 X}{dx^2} - \frac{1}{c^2} X(x) \frac{d^2 \Theta}{dt^2} = 0 \]

by \( u(x,t) = X(x) \Theta(t) \)

\[ \frac{1}{X(x)} \frac{d^2 X}{dx^2} - \frac{1}{c^2} \frac{d^2 \Theta}{dt^2} = 0 \]

Multiply throughout by \( c^2 \)

\[ \frac{c^2}{X(x)} \frac{d^2 X}{dx^2} - \frac{1}{\Theta(t)} \frac{d^2 \Theta}{dt^2} = 0 \]

\[ \frac{c^2}{X(x)} \frac{d^2 X}{dx^2} = \frac{1}{\Theta(t)} \frac{d^2 \Theta}{dt^2} = \text{constant} \]

The L.H.S. of above equation involves only independent variable \( x \) where as R.H.S. involves only independent variable \( t \). Since \( x \) and \( t \) are independent variables both the sides must be equal to some constant.

\[ \frac{c^2}{X(x)} \frac{d^2 X}{dx^2} = \frac{1}{\Theta(t)} \frac{d^2 \Theta}{dt^2} = -\omega^2 \]

In order to have oscillatory solution, the constant must be negative.

\[ \frac{c^2}{X(x)} \frac{d^2 X}{dx^2} = \frac{1}{\Theta(t)} \frac{d^2 \Theta}{dt^2} = -\omega^2 \]

If the R.H.S. is not a negative constant but a positive constant then the solution will not be oscillatory but will be exponentially decreasing or increasing function.

\[ \Rightarrow \frac{1}{\Theta(t)} \frac{d^2 \Theta(t)}{dt^2} = -\omega^2 \]

\[ \frac{c^2}{X(x)} \frac{d^2 X(x)}{dx^2} = -\omega^2 \]

Equation (2) can be solved as follows

\[ \frac{1}{\Theta(t)} \frac{d^2 \Theta(t)}{dt^2} = -\omega^2 \]

\[ \frac{d^2 \Theta(t)}{dt^2} + \omega^2(t) = 0 \]

This is 2\(^\text{nd}\) order, ordinary, linear different equation with constant coefficient.
\[ \Theta(t) = A \cos \omega t + B \sin \omega t \]

Equation no. (2) can also be solved similarly,

\[ \frac{c^2}{X(x)} \frac{d^2 X(x)}{dx^2} = -\omega^2 \]

\[ \frac{d^2 X(x)}{dx^2} + \left( \frac{\omega}{c} \right)^2 X = 0 \]

\[ \Rightarrow X(x) = c \cos \left( \frac{\omega}{c} x \right) + D \sin \left( \frac{\omega}{c} x \right) \]

\[ u(x,t) = X(x)\Theta(t) \]

\[ u(x,t) = (A \cos \omega t + B \sin \omega t \left( C \cos \left( \frac{\omega}{c} x \right) + D \sin \left( \frac{\omega}{c} x \right) \right)) \]

Applying the boundary conditions

i) \[ u(x,t) \bigg|_{x=0} = 0 = X(x) \bigg|_{x=0} \]

\[ \Rightarrow 0 = c \cos 0 + D \sin 0 = c \]

\[ c = 0 \]

\[ \Rightarrow X(x) = D \sin \left( \frac{\omega}{c} x \right) \]

ii) \[ u(x,t) \bigg|_{x=L} = 0 = X(x) \bigg|_{x=L} \]

\[ \Rightarrow D \sin \left( \frac{\omega L}{c} \right) = 0 \]

case i) \( D = 0 \).

This is not allowed as it gives the trivial solution that the string is at rest forever.

OR

Case ii)

\[ \sin \left( \frac{\omega L}{c} \right) = 0 \]

\[ \rightarrow \text{ case (a)} \quad \frac{\omega L}{c} = 0 \]

This is not allowed as \( \omega \neq 0 \) \( L \neq 0 \)

OR case (b) \( \frac{\omega L}{c} = n\pi \quad n = 1, 2, 3, \ldots \)

\[ \Rightarrow \omega_n = \frac{n\pi c}{L} \]

\[ \Rightarrow u(x,t) = X(x)\Theta(t) \]

\[ = D \sin \left( \frac{\omega x}{c} \right) \left[ A \cos \omega_n t + B \sin \omega_n t \right] \]

Without loss of any generality, the arbitrary constant \( D \) can be taken as unity.

\[ \therefore u(x,t) = A \sin \left( \frac{\omega x}{c} \right) \cos \omega_n t + B \sin \left( \frac{\omega x}{c} \right) \sin \omega_n t \]

\[ \therefore u_n(x,t) = A_n \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi c t}{L} \right) + B_n \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi c t}{L} \right) \quad n = 1, 2, 3, \ldots \quad n \in \mathbb{N} \]
Normal modes of oscillations:

The solution of the vibrating string are given by

\[ u_n(x,t) = A_n \sin \left( \frac{n \pi x}{L} \right) \cos \left( \frac{n \pi ct}{L} \right) + B_n \sin \left( \frac{n \pi x}{L} \right) \sin \left( \frac{n \pi ct}{L} \right) \]

If the value of \( n \) is fixed then the frequency of oscillation of the vibrating particle on the string is also fixed. But the frequency is same for all the particles on the string. Thus whenever, a string vibrates in such a way that each and every particle oscillates with the same frequency then this mode of vibration is called normal mode of vibrations.

It is observed that the number of normal modes of system of many particles is equal to the number of particles in the system. For e.g. two coupled harmonics oscillators has two particles. Hence, it has only two normal modes of oscillations namely symmetric and anti-symmetric modes of oscillation.

In the case of vibrating string, the number of particles is infinite. Therefore there are infinitely many normal modes of vibrations which can be obtained by permitting \( n = 1, 2, 3, \ldots \)

The most general solution of the vibrating string is given by,

\[ u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \]

\[ = \sum_{n=1}^{\infty} \left[ A_n \sin \left( \frac{n \pi x}{L} \right) \cos \left( \frac{n \pi ct}{L} \right) + B_n \sin \left( \frac{n \pi x}{L} \right) \sin \left( \frac{n \pi ct}{L} \right) \right] \]

All the arbitrary constants in the above general solution of the vibrating string can be obtained by using the following initial conditions.

i) \( u(x,t) \big|_{t=0} = u_0(x) \) \quad given

ii) \( \frac{\partial u(x,t)}{\partial t} \big|_{t=0} = v_0(x) \) \quad given

By using the Fourier Series Analysis, the constant \( A_n \) is given by,

\[ A_n = \frac{2}{L} \int_0^L u_0(x) \sin \left( \frac{n \pi x}{L} \right) dx \]

The constants \( B_n \) can be obtained similarly by differentially \( u \) with respect to time and then multiplying by \( \sin \left( \frac{n \pi x}{L} \right) \)

\[ B_n = \frac{2}{n \pi L} \int_0^L v_0(x) \sin \left( \frac{n \pi x}{L} \right) dx \]

Q. Show that if any function \( f(x-ct) \) satisfies the wave equation \( \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \) for small amplitude transverse waves on a stretched string then the wave velocity is \( \pm c \).

The function of the type \( f(x-ct) \) represents transverse waves on a stretched string as the waves travels along the string the phase at any point on the string is given by \( x-ct \).
In order to find the velocity of the waves we consider points on the string with the same phase.

\[ x - ct = \text{constant phase} \]

differentiating with respect to \( t \)

\[
\frac{dx}{dt} - c = 0
\]

\[
\frac{dx}{dt} = c
\]

This is the velocity of the waves traveling in the positive \( x \) direction. The velocity of the wave traveling in the opposite direction i.e. reflected waves is therefore \(-c\).

Hence the velocity of the waves can be \( \pm c \).

Let \( x - ct = y \)

\[
\frac{\partial f}{\partial x} = \frac{\partial f(x - ct)}{\partial x} = \frac{\partial f(y)}{\partial y} \frac{\partial y}{\partial x}
\]

\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left[ \frac{\partial f(y)}{\partial y} \frac{\partial y}{\partial x} \right]
\]

\[
= \frac{\partial}{\partial y} \left[ \frac{\partial f(y)}{\partial y} \frac{\partial y}{\partial x} \right] \frac{\partial y}{\partial x}
\]

\[
= \frac{\partial^2 f(y)}{\partial y^2} \left( \frac{\partial y}{\partial x} \right)^2
\]

\[
\therefore x - ct = y
\]

\[
\frac{\partial y}{\partial x} = 1
\]

\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}
\]

(1)

\[
\frac{\partial f}{\partial t} = \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
\]

\[
\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t} \left[ \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \right]
\]

\[
= \frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \right] \frac{\partial y}{\partial t}
\]

\[
= \frac{\partial^2 f(y)}{\partial y^2} \left( \frac{\partial y}{\partial t} \right)^2
\]

\[
\therefore \frac{\partial y}{\partial t} = -c
\]

\[
= \frac{\partial^2 f(y)}{\partial y^2} c^2
\]

\[
\frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial y^2}
\]

(2)

Comparing (1) and (2)
\[
\frac{\partial^2 f}{\partial x^2} = c^2 \frac{\partial^2 f}{\partial t^2}
\]

\[\Rightarrow f \text{ satisfies the wave equation}\]

2. Motion of string of length ‘L’ stretched between two points is given by

\[u(x,t) = \sum_{n=1}^{\infty} \left[ A_n \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi ct}{L} \right) + B_n \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi ct}{L} \right) \right]\]

If initially string is given transverse displacement at midpoint \(L/10\) from rest \((b,n)\) show that

i) \(A_n = 0\) for \(n = 2, 4, 6, \ldots\)

ii) \(B_n = 0\) for all values of \(n\)

\[u_0(x) = \begin{cases} \frac{1}{5}x & 0 \leq x \leq \frac{L}{2} \\ -\frac{1}{5}(x-L) & \frac{L}{2} \leq x \leq L \end{cases}\]

\[A_n = \frac{2}{L} \int_{0}^{L/2} u_0(x) \sin \left( \frac{n\pi x}{L} \right) \, dx\]

\[= \frac{2}{L} \int_{0}^{L/2} x \sin \left( \frac{n\pi x}{L} \right) \, dx + \frac{2}{L} \int_{L/2}^{L} (L-x) \sin \left( \frac{n\pi x}{L} \right) \, dx\]

\[= \frac{2}{5L} \int_{0}^{L/2} x \sin \left( \frac{n\pi x}{L} \right) \, dx + \frac{2}{5L} \int_{L/2}^{L} x \sin \left( \frac{n\pi x}{L} \right) \, dx + \frac{2}{5L} \int_{L/2}^{L} \left( L-x \right) \sin \left( \frac{n\pi x}{L} \right) \, dx\]

Consider

\[\int x \sin \left( \frac{n\pi x}{L} \right) \, dx = x \left[ \sin \left( \frac{n\pi x}{L} \right) - \frac{n\pi x}{L} \cos \left( \frac{n\pi x}{L} \right) \right] - \int \left[ \frac{n\pi x}{L} \cos \left( \frac{n\pi x}{L} \right) \right] \, dx\]

\[= x \left[ -x \cos \left( \frac{n\pi x}{L} \right) + \frac{L}{n\pi} \sin \left( \frac{n\pi x}{L} \right) \right] - \int \left[ \frac{L}{n\pi} \sin \left( \frac{n\pi x}{L} \right) \right] \, dx\]

\[= \frac{L}{n\pi} \left[ -x \cos \left( \frac{n\pi x}{L} \right) \right]_0^{L/2} + \left( \frac{L}{n\pi} \right)^2 \left[ \sin \left( \frac{n\pi x}{L} \right) \right]_0^{L/2}\]

\[= \frac{L}{n\pi} \left[ - \frac{L}{x} \cos \left( \frac{n\pi}{2} \right) \right] + \left( \frac{L}{n\pi} \right)^2 \left[ \sin \left( \frac{n\pi}{2} \right) \right]

Let \(n = 2m\) \(m = 1, 2, 3, \ldots\)

\[= \frac{L}{n\pi} \left[ - \frac{L}{x} \cos \left( \frac{n\pi}{2} \right) \right] + \left( \frac{L}{n\pi} \right)^2 \left[ \sin \left( \frac{n\pi}{2} \right) \right]\]

\[\int_{L/2}^{L} x \sin \left( \frac{n\pi x}{L} \right) \, dx\]
Let $n = 2m$, $m = 1, 2, 3$

$$= \frac{L}{n\pi} \left[ -x\cos\left(\frac{n\pi}{L}\right) \right] + \left( \frac{L}{n\pi} \right)^2 \left[ \sin\left(\frac{n\pi}{L}\right) \right]\frac{L}{2\pi m}$$

$$= \frac{L}{n\pi} \left[ -L\cos n\pi + \frac{L}{2}\cos\left(\frac{n\pi}{2}\right) \right] + \left( \frac{L}{n\pi} \right)^2 \left[ \sin n\pi - \sin \frac{n\pi}{L} \right]$$

**Kinematics of fluids**

The fluid is supposed to be divided into very large number of fluid elements called as fluid particles. The motion of fluid is characterised by certain parameters such as pressure, density, velocity etc. These parameters are functions of space and time co-ordinates. Laws of motion are applicable to actual fluid particles and not to the space co-ordinates. Therefore, we follow the fluid particle along its motion for small interval of time in order to find the total role of change of parameter with time. Consider the parameter pressure of the fluid.

$$dp = p(x + \hat{\delta}x, y + \hat{\delta}y, z + \hat{\delta}z, t + \hat{\delta}t) - p(x, y, z, t)$$

$$= \frac{\partial p}{\partial x} \hat{\delta}x + \frac{\partial p}{\partial y} \hat{\delta}y + \frac{\partial p}{\partial z} \hat{\delta}z + \frac{\partial p}{\partial t} \hat{\delta}t$$

$$= \frac{\partial p}{\partial t} + \left( \frac{i \partial p}{\partial x} + j \frac{\partial p}{\partial y} + k \frac{\partial p}{\partial z} \right) (\hat{i} \cdot v + j \cdot \hat{j} + k \cdot \hat{k})$$

$$= \frac{\partial p}{\partial t} + \hat{\nabla}p \cdot \hat{\nabla}p$$

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + \hat{\nabla}p \cdot \hat{\nabla}p$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \left( \hat{\nabla} \cdot \hat{\nabla}p \right)$$

To determine $\frac{d}{dt}(\partial V)$:

Consider a small volume element $\partial V$ is the form of an cubicle with the edges $\hat{\delta}x$, $\hat{\delta}y$, $\hat{\delta}z$.
We have
\[ \partial V = \partial x \partial \bar{y} \partial \bar{z} \]
\[
\frac{d}{dt} (\partial V) = \frac{d}{dt} (\partial x \partial \bar{y} \partial \bar{z})
\]
\[
= \left[ \frac{d}{dt} (\partial x) \right] \partial \bar{y} \partial \bar{z} + \left[ \frac{d}{dt} (\partial \bar{y}) \right] \partial x \partial \bar{z} + \left[ \frac{d}{dt} (\partial \bar{z}) \right] \partial x \partial y
\]
(1)

To find \( \frac{d}{dt} (\partial x) \)
\[
= [v_x]_{x=x_0} - [v_x]_0
\]
\[
= \lim_{\delta x \to 0} \left\{ [v_x]_{x=x_0} - [v_x]_0 \right\} \delta x
\]
\[
\frac{d (\partial x)}{dt} = \frac{\partial v_x}{\partial x} \delta x
\]
(2)

Similarly
\[
\frac{d}{dt} (\partial y) = \frac{\partial v_x}{\partial y} \delta y
\]
(3)
\[
\frac{d}{dt} (\partial z) = \frac{\partial v_x}{\partial z} \delta z
\]
(4)

Substituting (2), (3) and (4) in (1)
\[
\frac{d}{dt} (\partial V) = \left[ \frac{\partial v_x}{\partial x} \right] \partial \bar{y} \partial \bar{z} \partial x + \left[ \frac{\partial v_y}{\partial y} \right] \partial \bar{x} \partial \bar{z} \partial y + \left[ \frac{\partial v_z}{\partial z} \right] \partial \bar{x} \partial \bar{y} \partial z
\]
\[
= \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \partial V
\]
\[
\frac{d}{dt} (\partial V) = \nabla \cdot \partial V
\]

Characteristics of fluid motion:

1) Fluid flow can be viscous are non-viscous
2) The fluid flow can be streamline or turbulent
3) The fluid flow can be rotational or irrotational
4) The fluid can be compressible or incompressible

If the fluid is incompressible then rate of change of its volume will vanish
\[ \Rightarrow \frac{d}{dt}(\partial V) = 0 \]
\[ \Rightarrow \vec{\nabla} \cdot \vec{v} = 0 \]

The relation for the total time derivative for volume elements is given by
\[ \frac{d}{dt}(\partial V) = +\vec{\nabla} \cdot \vec{v} \partial V \]

This relation is true for all volume elements in different co-ordinate system. It is also valid for any finite volume of a fluid.

Consider the given finite volume \( V \) to be made up of large number of infinitesimal volume elements.

i.e. \( V = \sum_{} \partial V \)

Taking the summation of above relation
\[ \sum_{} \frac{d}{dt} \partial V = \sum_{} \left( + \vec{\nabla} \cdot \vec{v} \partial V \right) \]

L.H.S. = \[ \sum_{} \frac{d}{dt} \partial V \]
\[ = \frac{d}{dt} \sum_{} \partial V \quad \text{(Summation over volume element and time differentiation can be interchanged, as they are independent)} \]
\[ = \frac{dV}{dt} \]

R.H.S. = \[ \sum_{} \left( + \vec{\nabla} \cdot \vec{v} \partial V \right) \]
\[ = +\sum_{} \vec{\nabla} \cdot \vec{v} \partial V \]
\[ = +\int_{} \vec{\nabla} \cdot \vec{v} dV \]

\[ \therefore \frac{dV}{dt} = +\int_{} \vec{\nabla} \cdot \vec{v} dV \]

Applying Gauss' Divergence Theorem,
\[ \frac{dV}{dt} = +\int_{} \vec{v} \cdot ds \]
\[ = +\int_{} \vec{v} \cdot \hat{n} ds \]

Consider a small cap area ‘\( ds \)’ on the surface of given volume \( V \). it is imagined that the surface element \( ds \) moves out of the volume in a normal direction. The distance covered in time ‘\( dt \)’ is \( \vec{v} \cdot \hat{n} dt \). It is the length of the cylinder. Therefore, the volume of the fluid that goes out of this area in time ‘\( dt \)’ is \( (\vec{v} \cdot \hat{n} dt) ds \). Total volume of a fluid going out of a closed surface in time ‘\( dt \)’
\[ = \int_{} \vec{v} \cdot \hat{n} dt ds \]

But this must be equal to the change in a volume of the fluid in time \( dt \). Let it be ‘\( dV \)’.
\[ \int_{} \vec{v} \cdot \hat{n} dt ds = +dV \]
Therefore \( \frac{dV}{dt} = \oint \mathbf{v} \cdot \mathbf{n} ds \)

Physically it means that the rate of decrease of volume of a fluid must be the same as the rate at which the fluid escapes the surface enclosing the given volume. Thus is in agreement with conservation of fluid matter.

Imp.

Continuity equation for fluid flow:

Consider infinitesimal volume element \( \delta V \) of a given fluid. Let \( \rho \) be the density of a fluid.

\[ \text{.: Mass of the fluid element is given by} \]
\[ \delta m = \rho \delta V \]

According to conservation of mass principle, the net rate of change of mass should be zero.

i.e.
\[ \frac{d}{dt} (\delta m) = 0 \]
\[ \Rightarrow \frac{d}{dt} (\rho \delta V) = 0 \]
\[ \Rightarrow \frac{d \rho}{dt} \delta V + \rho \frac{d(\delta V)}{dt} = 0 \]

Using identities

i) \[ \frac{d \rho}{dt} = \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho \]

\[ \Rightarrow \left[ \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho \right] \delta V + \rho \left[ + \mathbf{\nabla} \cdot \mathbf{v} \delta V \right] = 0 \]

\[ \Rightarrow \frac{\partial \rho}{\partial t} + \mathbf{\nabla} \cdot (\rho \mathbf{v}) = 0 \]

This is required continuity equation.

This is in agreement with principle of conservation of mass. This can be explained as follows.

\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \]

Integrating over a volume \( V \)

\[ \int \frac{\partial \rho}{\partial t} dV = -\int \mathbf{\nabla} \cdot (\rho \mathbf{v}) dV \]

\[ \frac{\partial}{\partial t} \int_{V} \rho dV = -\int_{\partial V} (\rho \mathbf{v}) \cdot \mathbf{n} dS \quad \text{(using Gauss’ divergence theorem)} \]

\[ \Rightarrow \frac{\partial}{\partial t} (\text{mass}) = -\oint (\rho \mathbf{v}) \cdot \mathbf{n} ds \]
The L.H.S. of the above equation represents the rate of change of mass of the fluid in the given volume $V$ whereas R.H.S. represents the rate of flow of mass of the fluid through the closed surface ‘S’ bounding the given volume $V$

Imp.

**Physical Meaning of curl of $\vec{v}$:** $\nabla \times (\vec{v} \times \vec{v})$

Consider a fluid motion containing a vortex. Let $V$ be the fluid velocity around the vortex. Consider the surface integration of curl of $\vec{v}$ (i.e. $(\vec{v} \times \vec{v})$)

$$\int_{\text{surf}} (\vec{v} \times \vec{v}) \cdot \hat{n} ds$$

Using Stoke’s Theorem we get

$$\int_{\text{surf}} (\vec{v} \times \vec{v}) \cdot \hat{n} ds = \oint_{\text{C}} \vec{v} \cdot d\vec{l} \neq 0$$

**Fluid motion with vortex**

The closed path i.e. the boundary of surface is a circular path. At each and every point on this path the velocity of the fluid and the time element ‘$dt$’ have the same direction. Therefore $\vec{v} \cdot d\vec{l}$ is never zero. This implies that the integration not equal to zero. In turn, it implies that the integrand of the first integration is not zero. Thus $\vec{v} \times \vec{v} \neq 0$

In other words, whenever there is a rotational motion of the fluid, curl of $\vec{v}$ i.e. $(\vec{v} \times \vec{v})$ is non-zero. Thus $\vec{v} \times \vec{v}$ is a measure of rotational motion or curling of the fluid or angular velocity of the fluid motion.

Consider a general case in which apparently, there is no vortex in the fluid flow but there is velocity gradient. In such a situation, it is observed that $\vec{V} \times \vec{v} \neq 0$. Even in this case it can be shown that $(\vec{V} \times \vec{v})$ represents the angular velocity of the fluid motion.

A rotating co-ordinate system rotating with constant angular velocity $\vec{\omega}$. The velocity of the fluid in inertial frame and that in the rotating frame are related by $\vec{v} = \vec{v}^* + \vec{\omega} \times \vec{r}$

Where $\vec{v} \rightarrow$ fluid velocity in inertial frame
$\vec{v}^* \rightarrow$ fluid velocity in rotating frame
\( \dot{\omega} \rightarrow \text{angular velocity of rotating frame} \)
\( \vec{r} \rightarrow \text{radius vector} \).

Taking curl of the above relation
\[
\vec{\nabla} \times \vec{v} = \vec{\nabla} \times \vec{v}' + \vec{\nabla} \times (\dot{\omega} \times \vec{r})
\]

Using product rule we get,
\[
\vec{\nabla} \times \vec{v} = \vec{\nabla} \times \vec{v}' + \dot{\omega}(\vec{\nabla} \cdot \vec{r}) - \vec{r}(\vec{\nabla} \cdot \dot{\omega}) + (\vec{r} \cdot \vec{\nabla})\dot{\omega} - (\dot{\omega} \cdot \vec{r})\vec{r}
\]

Since \( '\omega' \) is constant we get,
\[
\vec{\nabla} \times \vec{v} = \vec{\nabla} \times \vec{v}' + 3\dot{\omega} - \dot{\omega}
\]
\[
= \vec{\nabla} \times \vec{v}' + 2\dot{\omega}
\]

Since \( \dot{\omega} \) is angular velocity of a rotating frame that we have introduced, we select, \( \dot{\omega} = \frac{1}{2} \vec{\nabla} \times \vec{v} \)
\[
\Rightarrow \vec{\nabla} \times \vec{v} = \vec{\nabla} \times \vec{v}' + \vec{\nabla} \times \vec{v}
\]
\[
\Rightarrow \vec{\nabla} \times \vec{v}' = 0
\]

Thus this means that in rotating frame fluid has no angular velocity. In other words, fluid has angular velocity in the inertial frame. Thus, \( \vec{\nabla} \times \vec{v} \) is still the measure of angular velocity of the fluid motion.

**Equation of motion for an ideal fluid:**

The equation of motion is applicable to only ideal fluid motion. The fluid motion is called ideal if it does not support any shearing stress. In other words, the fluid motion is due to pressure difference only or any other external body force. If fluid does not support any shearing stress, it means the fluid has zero or negligible viscosity.

Consider small volume element \( \delta V \) in the form of parallelepiped.

We calculate net force acting on the volume element due to the pressure only. It is assumed that fluid moves along +ve X-direction. Then the pressure of \( x + \delta x \) is less than that at \( x \). Hence the net force acting on the volume element along the X-direction due to pressure difference is given by

\[
\delta F_x = (\text{force})_x - (\text{force})_{x+\delta x}
\]
\[
= (p \frac{\partial y}{\partial x} \delta x)_x - (p \frac{\partial y}{\partial x} \delta x)_{x+\delta x}
\]
\[
= \lim_{\delta x \to 0} \left[ \left( p \frac{\partial y}{\partial x} \right)_x - (p \frac{\partial y}{\partial x})_x \right] \delta x
\]
\[
\delta F_x = \left( \frac{\partial p}{\partial x} \right) \delta x \frac{\partial y}{\partial x} \delta x \quad (1)
\]

Similarly,
\[ \delta F_y = - \left( \frac{\partial p}{\partial y} \right) \delta x \delta y \delta z \]  \hspace{1cm} (2)

\[ \delta F_z = - \left( \frac{\partial p}{\partial z} \right) \delta x \delta y \delta z \]  \hspace{1cm} (3)

\[ \therefore \text{Net force acting on the volume element due to pressure difference is given by,} \]

\[ \delta F = \delta F_x i + \delta F_y j + \delta F_z k \]

\[ = \left( \frac{\partial p}{\partial x} i + \frac{\partial p}{\partial y} j + \frac{\partial p}{\partial z} k \right) \delta x \delta y \delta z \]

\[ \delta F = \partial F + \delta F_x + \delta F_y + \delta F_z \]

\[ \delta F = - \nabla p \delta V \]  \hspace{1cm} (4)

In addition to this force due to pressure, there may be additional external force acting on the body of a fluid element. Let \( \vec{f} \) this external body force acting on the fluid per unit volume. Therefore, the body force acting on the volume element \( \delta V \) will be \( \int \vec{f} \delta V \)

According to Newton’s 2nd law of motion the product of mass and acceleration of the fluid element must be equal to net force acting on the element

\[ \text{(mass)} \times \text{(acceleration)} = \text{Net force} \]

\[ (\rho \delta V) \left( \frac{d\vec{V}}{dt} \right) = -\nabla p \delta V + \int \vec{f} \delta V \]

\[ \delta V \text{ is an arbitrary volume element} \neq 0 \]

\[ \therefore \rho \frac{d\vec{V}}{dt} = -\nabla p + \int \vec{f} \]

\[ (f \text{ may be gravitational force}) \]

\[ \therefore \rho \left( \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right) = -\nabla p + \int \vec{f} \]

\[ \therefore \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} + \frac{1}{\rho} \nabla p = \int \vec{f} \]

This is required equation of motion for the ideal fluid motion. It is known as Euler’s equation of motion for the ideal fluid.

**Conservation Laws in Fluid Motion:**

There are several conservation laws applicable to the system of particles such as conservation of mass, conservation of linear momentum, conservation of angular momentum and conservation of energy. All these conservation laws are because of Newton’s laws of motion.

Similar conservation laws are applicable even for fluid motion. For e.g. Continuity equation for fluid motion leads to conservation at mass principle.

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \]

This continuity equation can be generalised which will take into account a source of fluid. Let \( Q \) be the rate of generation of fluid per unit volume. In this case, the continuity equation modifies to \( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = Q \)

Multiplying by \( dV \) and integrating over volume \( dV \); \( \int \frac{\partial \rho}{\partial t} \delta V + \int_{\text{surf}} (\rho \vec{v}) \cdot \hat{n} ds = \int_{\text{vol}} Q dV \)
The above equation implies that rate of increase of fluid mass plus rate of escape of mass through the surface bounding the volume must be the same as the rate at which the mass of fluid is generated in the volume.

**Conservation of linear momentum:**
In fluid motion, the product \( \rho v \) plays the same role as the linear momentum in particle dynamics. It is termed as linear momentum density.

Consider Euler’s equation:

\[
\rho \left( \frac{\partial v}{\partial t} + \nabla p \right) = \vec{f} \delta V
\]

Integrate over volume \( V \)

\[
\frac{d}{dt} \int (\rho \vec{v}) dV = -\int \vec{v} \nabla p dV + \int \vec{f} dV
\]

Using Gauss’s theorem

\[
= \int \vec{p} \cdot \hat{n} ds + \int \vec{f} dV
\]

According to linear momentum conservation principle for particle dynamics, rate of change of momentum is always zero or momentum is conserved if no external force acts on the system.

In case of fluid motion external force is \( \vec{f} \). We assume \( \vec{f} = 0 \).

\[
\frac{d}{dt} \int (\rho \vec{v}) dV = \int \vec{p} \cdot \hat{n} ds
\]

The first term on R.H.S. of the above equation is the force acting on the bounding surface due to the pressure exerted by the fluid outside the volume. Thus within the volume the rate of change of linear momentum of the fluid is always zero. This is the conservation of linear momentum.

Similarly, conservation of angular momentum for the fluid can be established.

Oct.2000

Consider a fluid flow in which velocity \( \vec{v}(x,t) = \frac{at}{x} \hat{i}, x > 0 \) Find acceleration \( \vec{a}(x,t) \) of fluid element at position \( x \) and time ‘\( t \)’. Is the fluid flow incompressible? Explain.

\[
\vec{a}(x,t) = \frac{d\vec{v}}{dt} = \frac{d}{dt} \left( \frac{at}{x} \right) \hat{i}
\]

Using \( \frac{d}{dt} \left| \vec{v} \right| = \frac{\partial}{\partial t} \left| \vec{v} \right| + \left( \vec{v} \cdot \nabla \right) \left| \vec{v} \right| \)

\[
\vec{a}(x,t) = \frac{\partial}{\partial t} \left| \vec{v} \right| + \left( \vec{v} \cdot \nabla \right) \left| \vec{v} \right|
\]

\[
= \frac{\partial}{\partial t} \left( \frac{at}{x} \right) \hat{i} + \left( \frac{at}{x} \right) \left( \frac{\partial}{\partial x} \left( \frac{at}{x} \right) \hat{i} + \frac{\partial}{\partial y} \left( \frac{at}{x} \hat{j} + \frac{\partial}{\partial z} \left( \frac{at}{x} \hat{k} \right) \right) \right) \frac{at}{x}
\]

\[
= \frac{a \hat{i}}{x} + \frac{at}{x} \frac{\partial}{\partial x} \left( \frac{at}{x} \right) \hat{i}
\]
\[ \frac{a}{x} + \left( \frac{at}{x} \right) - \frac{at}{x^2} i \]
\[ \frac{a}{x} \left( \frac{a^2 t^2}{x^3} \right) i \]

The fluid is said to be incompressible if \( \nabla \cdot \vec{v} = 0 \)

Consider \( \nabla \cdot \vec{v} = \left( \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial \hat{z}} \right) \left( \frac{at}{x} \right) i \)
\[ \frac{\partial}{\partial x} \left( \frac{at}{x} \right) i \]

as \( x > 0 \) and \( t \geq 0 \)
\[ \therefore \nabla \cdot \vec{v} \neq 0 \] The fluid is compressible

V.Imp.

String as a limiting case of system of particles:
Consider finite number of particles connected between two fixed ends. The first particle and the last particle are at rest always. We assume that these particles move only in vertical direction and there is absolutely no sidewise deflection. The amplitude of vibration is assumed small for all the particles. There is constant separation between the particles (They are equally spaced.)

Let \( \tau \) be the constant tension with which the particles are acted upon by adjacent particles. Let \((N+2)\) be the total number of particles with first and the last at rest. The distance between the first and last particle i.e. the length of the string is, \((n+1)h = L.\n
Let mass of each particle be same say ‘m’ thus the equation of motion for the system of \(N\) particles is given by

\[ m \frac{d^2 u_j}{dt^2} = \tau \left( \frac{u_{j+1} - u_j}{h} \right) - \tau \left( \frac{u_j - u_{j-1}}{h} \right) \]

Where \( u_j \) is the displacement of the \(j^{th}\) particle along vertical direction i.e. \( u(jh, \hat{y}) = u_j \).

In this case linear mass density is
\[ m(n+1)h = \frac{m}{(n+1)h} \]

The above equation of motion can be simplified as
\[ \frac{d^2 u_j}{dt^2} = \left( \frac{\tau}{\rho} \right) \frac{1}{h} \left[ \frac{u_{j+1} - u_j}{h} - \frac{u_j - u_{j-1}}{h} \right] \]

We have in the limit \( h \) becoming very small
\[ \left[ \frac{\partial u}{\partial x} \right]_{j+\frac{1}{2}h} = \lim_{h \to 0} \left[ \frac{u_{j+1} - u_j}{h} \right] \]
\[ \left[ \frac{\partial u}{\partial x} \right]_{j-\frac{1}{2}h} = \lim_{h \to 0} \left[ \frac{u_j - u_{j-1}}{h} \right] \]
\[
\frac{\partial^2 u_j}{\partial t^2} = \left( \frac{\tau}{\sigma} \right) \frac{1}{h} \left[ -\frac{\partial u}{\partial x_j} \right] \left[ \frac{\partial u}{\partial x_j} \right] \left[ \frac{\partial^2 u}{\partial x_j^2} \right] 
\]

\[
= \frac{\tau}{\sigma} \lim_{h \to 0} \left[ \frac{1}{h} \left[ -\frac{\partial u}{\partial x_j} \right] \left[ \frac{\partial u}{\partial x_j} \right] \left[ \frac{\partial^2 u}{\partial x_j^2} \right] \right] 
\]

\[
\frac{\partial^2 u_j}{\partial t^2} = \frac{\tau}{\sigma} \frac{\partial^2 u}{\partial x_j^2} 
\]

In the limiting case when \( h \to 0 \) or the number of particles tends to infinity, we get the same differential equation with continuous distribution of mass along the string.

**Conservation of energy**

We have Euler's equation

\[
\rho \frac{d\vec{V}}{dt} \cdot \vec{V} + \vec{V} \cdot \rho \frac{d\vec{V}}{dt} = \vec{f} \cdot \vec{V} 
\]

\[
\frac{d}{dt} (\rho \vec{V} \cdot \vec{V}) + \vec{V} \cdot \rho \frac{d\vec{V}}{dt} - \vec{f} \cdot \vec{V} = 0 
\]

But \( \rho \vec{V} = \delta m \) (constant) taking dot product with fluid velocity \( \vec{v} \) we get,

\[
\vec{v} \cdot \frac{d}{dt} \left( \frac{1}{2} \rho \vec{V} \cdot \vec{V} \right) + \vec{V} \cdot \rho \frac{d\vec{V}}{dt} - \vec{f} \cdot \vec{V} = 0 
\]

\[
\therefore \frac{d}{dt} \left( \frac{1}{2} \rho \vec{V} \cdot \vec{V} \right) + \vec{V} \cdot \rho \frac{d\vec{V}}{dt} - \vec{f} \cdot \vec{V} = 0 
\]

\[
(1) 
\]

Consider

\[
\frac{d}{dt} (\rho \vec{V}) = \frac{dp}{dt} \vec{V} + \rho \frac{d}{dt} (\vec{V}) 
\]

\[
= \dot{\rho} \vec{V} + \vec{V} \cdot \rho \frac{d\vec{V}}{dt} + \rho \vec{V} \cdot \vec{V} 
\]

\[
\therefore \vec{V} \cdot \rho \frac{d\vec{V}}{dt} = \frac{d}{dt} (\rho \vec{V}) - \dot{\rho} \vec{V} - \rho \vec{V} \cdot \vec{V} 
\]

\[
(2) 
\]

Generally the body force acting on the fluid is gravitational force. Therefore,

\[
\vec{f} \cdot \vec{V} = \delta \vec{g} 
\]

\[
= \rho \delta \vec{V} \cdot \vec{g} 
\]

Any vector field can always be expressed in the form of a scalar potential function.

For e.g. electric field is given by,

\[
\vec{E} = -\vec{V} \cdot \vec{g} 
\]

Where \( V \) = scalar potential function

Similarly the intensity of gravitational field can be expressed in terms of gravitational potential function \( G \).

\[
\therefore \vec{g} = \vec{V} G 
\]

Where, \( G \) is gravitational potential function

Consider

\[
-\vec{V} \cdot \vec{g} \delta V = -\vec{V} \cdot \rho \delta \vec{V} G 
\]

\[
= -(\vec{V} \cdot \vec{V} G) \delta V 
\]
We have, by definition
\[ \frac{dG}{dt} = \frac{\partial G}{\partial t} + \vec{v} \cdot \nabla G \]
\[- \vec{v} \cdot \nabla G = \frac{\partial G}{\partial t} - \frac{dG}{dt} \]
\[ : - \vec{v} \cdot \vec{f} \delta V = \left( \frac{\partial G}{\partial t} - \frac{dG}{dt} \right) \rho \delta V \quad \text{(3)} \]
Substituting from equation (2) and (3) in equation (1) we get,
\[ \frac{d}{dt} \left( \frac{1}{2} \rho \delta V^2 \right) + \frac{d}{dt} \left( \rho \vec{v} \rho \delta V - p \vec{v} \rho \delta \vec{v} + \left( \frac{\partial G}{\partial t} - \frac{dG}{dt} \right) \rho \delta V \right) = 0 \]
\[ \frac{d}{dt} \left( \frac{1}{2} \rho \delta V^2 + p \delta V - p \delta V \right) = \frac{\partial \rho}{\partial t} \delta V - p \vec{v} \cdot \vec{v} \delta V - \frac{\partial G}{\partial t} \rho \delta V \]

Now,
a) \[ \frac{\partial G}{\partial t} = 0 \] Because, the gravitational potential function does not depend explicitly on time
b) \[ \frac{\partial p}{\partial t} = 0 \] Generally, we consider steady flow, therefore the fluid pressure may change from place to place but at a given point pressure is independent of time.
c) \[ \vec{V} \cdot \vec{v} = 0 \] For incompressible fluids
\[ : \frac{1}{2} \rho \delta V^2 \] is the K.E. of the fluid element \( \delta V \), \( \rho \delta V \) is the P.E. of the element due to pressure and \( G\delta \rho \delta \) is the gravitational potential energy of the fluid.
Thus, for steady and incompressible fluid flow we have,
\[ \frac{d}{dt} \left[ \frac{1}{2} \rho \delta v^2 + p - \rho G \right] = 0 \]
\[ \Rightarrow \frac{1}{2} \rho \delta v^2 + p - \rho G = \text{constant} \]
This is the conservation of energy principle. This itself is known as Bernoulli’s equation for ideal fluid.

Ex. 2001

Moving Co-ordinate System

1) Moving origin of co-ordinates:-

By triangle law of vector
\[ \vec{r} = \vec{r}' + \vec{R} \]
Differentiating with respect to ‘\( t \)’
\[
\frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \frac{d\vec{R}}{dt}
\]

Differentiating with respect to \(\phi\)

\[
\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}'}{dt^2} + \frac{d^2\vec{R}}{dt^2}
\]

\(\vec{a} = \vec{a}' + \vec{a}_g\)

If starred frame is with uniform (constant) velocity with respect to unstarred frame then,
\(\vec{a}_g = 0\)
\(\vec{a} = \vec{a}'\)

\(\vec{m} = \vec{m}'\)
\(\vec{m} = \vec{F} \quad \text{Newton’s second law of motion}\)
\(\Rightarrow \vec{m}' = \vec{F}\)

An inertial frame is defined as the frame in which Newton’s laws of motion are valid. All the frames moving with constant velocity with respect to each other are all equivalent inertial frames. This is Newtonian principle of relativity.

If \(\vec{a}_g \neq 0\)
Then in (S) frame
\(\vec{m} = \vec{F} \quad \text{Newton’s laws of motion}\)

But in (S’) frame
\(\vec{m}' \neq \vec{F} \quad \text{Newton’s laws not applicable}\)

\(S’\) is called as non-inertial frame of reference because it has acceleration with respect to \(S\) frame.

\(\vec{m} = m(\vec{a}' + \vec{a}_g)\)
\(\vec{m}' = \vec{m} - \vec{m}_g\)
\(\vec{m}' = \vec{F} - \vec{m}_g\)

Imp.
1) Rotating co-ordinate system

In (S) frame,
Position vector of p is \(\vec{r} = \hat{x}i + \hat{y}j + \hat{z}k\)
In (S’) frame,
\(\vec{r} = \hat{i}'i' + \hat{j}'j' + \hat{k}'k'\) \hspace{1cm} (Position vector need not be \(r\) since it is distance from origin and \(O\) and \(O'\) always coincide)

\(i', j', k'\) are functions of \(i, j, k\) (since they are rotating is any arbitrary direction)
Our aim is to relate the velocities of the particle as measured in two frames $S$ and $S^*$

By definition,

$$\frac{d\vec{B}}{dt} = \lim_{\Delta t \to 0} \left\{ \frac{\vec{B}(t + \Delta t) - \vec{B}(t)}{\Delta t} \right\} = \frac{\text{distance covered in time } \Delta t}{\text{Time taken } \Delta t}$$

But in same time $\Delta t$, vector will travel a different distance in in two different frames.

$$\left( \frac{d}{dt} \right) \to \text{velocity with respect to } S$$

$$\left( \frac{d^*}{dt} \right) \to \text{velocity with respect to } S^*$$

$$\Rightarrow \frac{d^*}{dt} = \frac{d}{dt} \{ x\dot{i} + y\dot{j} + z\dot{k} \}$$

In $S^*$ frame,

$$\frac{d^*\vec{r}}{dt} = \frac{d}{dt} \{ x^*\dot{i}^* + y^*\dot{j}^* + z^*\dot{k}^* \}$$

Consider, velocity of a particle of position vector $\vec{r}$ with respect to $(S)$ frame. But the position vector $\vec{r}$ is expressed in terms of co-ordinates of $(S^*)$ frame. In this case, even the unit vectors $\dot{i}^*, \dot{j}^*$ etc. are not at rest with respect to $(S)$ frame.

Therefore

$$\frac{d\vec{r}}{dt} = \frac{d}{dt} \{ x\dot{i} + y\dot{j} + z\dot{k} \}$$

$$= \frac{d}{dt} \left[ x^*\dot{i}^* + y^*\dot{j}^* + z^*\dot{k}^* \right]$$

$$= \left\{ \frac{dx^*}{dt}\dot{i}^* + \frac{dy^*}{dt}\dot{j}^* + \frac{dz^*}{dt}\dot{k}^* \right\} + \left\{ x^*\frac{d\dot{i}^*}{dt} + y^*\frac{d\dot{j}^*}{dt} + z^*\frac{d\dot{k}^*}{dt} \right\}$$

The time derivatives of unit vectors of $S^*$ frame with respect to $(S)$ frame can be obtained as follows:

Let vector $\vec{B}$ be any vector at rest in the frame $S^*$. It’s time derivative with respect to $(S)$ frame is as usual given by

$$\frac{d\vec{B}}{dt} = \lim_{\Delta t \to 0} \left\{ \frac{\vec{B}(t + \Delta t) - \vec{B}(t)}{\Delta t} \right\}$$

$$\Delta \vec{B} = \vec{B}(t + \Delta t) - \vec{B}(t)$$
\[
\begin{align*}
\Delta \mathbf{B} &= (B \sin \theta) \omega \Delta t \\
\frac{\Delta \mathbf{B}}{\Delta t} &= \omega B \sin \theta
\end{align*}
\]

Even in direction it becomes

\[
\frac{\Delta \mathbf{B}}{\Delta t} = \ddot{\mathbf{B}}
\]

\[
\lim_{\Delta t \to 0} \frac{\Delta \mathbf{B}}{\Delta t} = \frac{d\mathbf{B}}{dt} = \ddot{\mathbf{B}}
\]

\(\mathbf{B}\) is supposed to be at rest in \(S^*\) frame. Similarly, unit vectors \(\hat{i}^*, \hat{j}^*, \hat{k}^*\) are at rest in \(S^*\) frame. Therefore, we have,

\[
\begin{align*}
\frac{d}{dt}(\hat{i}^*) &= \ddot{\omega} \times \hat{i}^* \\
\frac{d}{dt}(\hat{j}^*) &= \ddot{\omega} \times \hat{j}^* \\
\frac{d}{dt}(\hat{k}^*) &= \ddot{\omega} \times \hat{k}^*
\end{align*}
\]

Using this result, we get

\[
\begin{align*}
\frac{d\mathbf{r}}{dt} &= \frac{d\mathbf{r}^*}{dt} + x^*(\ddot{\omega} \times \hat{i}^*) + y^*(\ddot{\omega} \times \hat{j}^*) + z^*(\ddot{\omega} \times \hat{k}^*) \\
&= \frac{d\mathbf{r}^*}{dt} + \ddot{\omega} \times (x^* \hat{i}^* + y^* \hat{j}^* + z^* \hat{k}^*) \\
\frac{d^2\mathbf{r}}{dt^2} &= \frac{d^2\mathbf{r}^*}{dt^2} + \ddot{\omega} \times \frac{d\mathbf{r}}{dt}
\end{align*}
\]

Velocity of particle with respect to \((S)\) frame is usual to the sum of the velocity of the same particle with respect to \(S^*\) frame and \(\ddot{\omega} \times \mathbf{r}\)

The acceleration of a particle with respect to the frames \((S)\) and \(S^*\) frame and relation between them can be obtained as follows:

\[
\begin{align*}
\frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt} \left( \frac{d\mathbf{r}}{dt} \right) \\
&= \frac{d}{dt} \left( \frac{d\mathbf{r}^*}{dt} + \ddot{\omega} \times \mathbf{r} \right) \quad \text{using (1)} \\
&= \frac{d}{dt} \left( \frac{d\mathbf{r}^*}{dt} \right) + \frac{d}{dt} (\ddot{\omega} \times \mathbf{r}) \\
&= \left[ \frac{d^2\mathbf{r}^*}{dt^2} + \ddot{\omega} \times \frac{d\mathbf{r}}{dt} \right] + \frac{d\ddot{\omega}}{dt} \times \mathbf{r} + \ddot{\omega} \times \frac{d\mathbf{r}}{dt} \\
&= \frac{d^2\mathbf{r}^*}{dt^2} + \ddot{\omega} \times \frac{d\mathbf{r}}{dt} + \left[ \frac{d\ddot{\omega}}{dt} + \ddot{\omega} \times \mathbf{r} \right] \times \mathbf{r} + \ddot{\omega} \times \left[ \frac{d\mathbf{r}}{dt} \right] + \ddot{\omega} \times \mathbf{r}
\end{align*}
\]
This equation gives a relation between accelerations of a particle with respect to
the frame (S) and acceleration with respect to frame S* where S* rotates with
angular velocity \( \omega \) with respect to (S) frame. The above relation is called
coriolis theorem.

\[ \ddot{\omega} \times (\dot{\omega} \times \vec{r}) \] is called centripetal acceleration

\[ 2\dot{\omega} \times \frac{d\vec{r}}{dt} \] is called coriolis acceleration

\[ \frac{d\dot{\omega}}{dt} \times \vec{r} \] is called has no specific name

It is non-zero when even rotating frame has acceleration.

Remarks:

1) C.P. acceleration = \( \ddot{\omega} \times (\dot{\omega} \times \vec{r}) \)

\[ |\ddot{\omega} \times (\dot{\omega} \times \vec{r})| = \epsilon \dot{\omega} \times \vec{r} \sin 90^\circ \]
= \( \omega (\omega r \sin \theta) \)
= \( \omega^2 r \sin \theta \)

2) Coriolis acceleration. = \( 2\dot{\omega} \times \frac{d\vec{r}}{dt} \)

Carioles acceleration is present only when particle has non-zero velocity and
not parallel to axis of rotation with respect to S* frame.

3) The term \( \frac{d\dot{\omega}}{dt} \times \vec{r} \)
It is present only if S* has angular acceleration with respect to S frame.

Equation of motion of a particle near the surface of earth [Is an
earth’s frame is inertial frame:]
Co-ordinate system fixed with respect to earth is non inertial frame because
earth rotates about an axis passing through south and north poles with
constant angular velocity.
Consider a particle of mass ‘m’ experiencing force \( F \) in addition to the
gravitational force acting on the particle. If S is an inertial frame fixed with
respect to star then according to Newton’s law of motion we have,

\[ m \frac{d^2\vec{r}}{dt^2} = \vec{F} + m\vec{g} \quad (1) \]
According to Coriolis Theorem the relation between acceleration of a particle in 
(S) and (S') frame is given by,
\[
\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}'}{dt^2} + \mathbf{\omega} \times \mathbf{r}' + 2\mathbf{\omega} \times \frac{d\mathbf{r}'}{dt} + \frac{d\mathbf{\omega}}{dt} \times \mathbf{r}'
\]

Where \( S' \) is a rotating frame fixed with respect to earth and origin corresponding with centre of the earth. It is rotating with constant angular velocity \( \mathbf{\omega} \). \( \therefore \frac{d\mathbf{\omega}}{dt} = 0 \)
\[
\therefore \frac{d^2\mathbf{r}'}{dt^2} = \frac{d^2\mathbf{r}'}{dt^2} + \mathbf{\omega} \times \mathbf{r}' + 2\mathbf{\omega} \times \frac{d\mathbf{r}'}{dt}
\]

Multiplying both sides by \( m \)
\[
m \frac{d^2\mathbf{r}'}{dt^2} + m\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}') + 2m\mathbf{\omega} \times \frac{d\mathbf{r}'}{dt} = \mathbf{F} + mg
\]
\[
m \frac{d^2\mathbf{r}'}{dt^2} = \mathbf{F} + mg - m\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}') - 2m\mathbf{\omega} \times \frac{d\mathbf{r}'}{dt}
\]

Where \( \frac{d^2\mathbf{r}'}{dt^2} \) is acceleration of the particle with respect to the \( S' \) frame is earth’s frame.

This is the required equation of motion near the surface of the earth.

\( m\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}') \) is called centrifugal force

\( 2m\mathbf{\omega} \times \left( \frac{d\mathbf{r}'}{dt} \right) \) is called coriolis force

[diagram]

Centrifugal acceleration = \( |\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}')| = \mathbf{\omega} \times \mathbf{r}' \sin \theta \)

At equator, \( \theta = 90^\circ \)

Centrifugal acceleration is maximum
\[
\omega \times \mathbf{r}' = \left( \frac{2\pi}{T} \right)^2 \mathbf{r} - \left( \frac{2\pi}{24 \times 3600} \right)^2 R = 3.37 cm/ s^2
\]

At poles \( \theta = 0^\circ \) Centrifugal acceleration =0 and no centrifugal force

\[
m \frac{d^2\mathbf{r}'}{dt^2} = \mathbf{F} + mg - m\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}') - 2m\mathbf{\omega} \times \frac{d\mathbf{r}'}{dt}
\]

At poles \( g_{\text{eff}} \) max = \( g \)

At Equator \( g_{\text{eff}} \) max = \( g - \mathbf{\omega} \times \mathbf{r}' = g - \omega^2 R \)

This is the reason why the earth is flattened at the poles and bulging at the equator.
Let $\phi$ be the angle between real vertical and apparent vertical.

$$\tan \phi \approx \phi$$

**Horizontal component of centrifugal acceleration**

$$\text{Vertical component of acceleration}$$

$$= \frac{\omega^2 r \sin \theta \cos \theta}{g} - \omega^2 r \sin^2 \theta$$

$$= \frac{\omega^2 r \sin \theta \cos \theta}{g}$$

$$\approx \frac{\omega^2 r \sin \theta \cos \theta}{g} \quad \because \text{2nd term is vanishingly small}$$

$$= \frac{\omega^2 r \sin 2\theta}{2g}$$

$$= \frac{\omega^2 r}{2g} \quad \theta = 45^\circ$$

$$\approx 0^\circ 6'$$

The coriolis force $2m\vec{\omega} \times \frac{d\vec{r}}{dt}$ is effective whenever particle on the surface of the earth has non-zero velocity with respect to the earth.

There are three observable effects of coriolis force:

1) A particle dropped vertically down is observed to shift towards the east side due to coriolis force.

2) Air mass rushes to an area of low pressure. This air mass is observed to shift towards right due to coriolis force. This leads to formation of Cyclones. They are anticlockwise in northern hemisphere and clockwise in southern hemisphere.

3) Plane of oscillation of a simple pendulum is observed to rotate due to coriolis force e.g. Foucault’s Pendulum

Problems:

1) Calculate the magnitude of Coriolis Acceleration of a particle of moving with velocity 200m/s at an angle of 60° with the angular velocity $\omega$ of rotation of Earth.

$$\text{Coriolis Acceleration} = \left| 2\vec{\omega} \times \frac{d\vec{r}}{dt} \right|$$

$$= 2\omega \left| \frac{d\vec{r}}{dt} \right| \sin 60^\circ$$
\[ \omega = 7.273 \times 10^{-5} \text{rad/dec} \quad \omega = \left( \frac{2\pi}{T} \right) \]

Coriolis Acceleration
\[ = 2 \times 7.273 \times 10^{-5} \times 200 \times \sin 60 \]
\[ = 2.52 \text{cm/s}^2 \]

2) Calculate the centripetal acceleration of a particle at a place where co-latitude is 30° and radius of earth 6400km

Centripetal Acceleration
\[ \left[ \ddot{\omega} \times (\ddot{\omega} \times \vec{r}) \right] \]
\[ = \omega^2 r \sin \theta \]
\[ = (7.273 \times 10^{-5})^2 \times (6.4 \times 10^6) \sin 30 \]
\[ = 1.69 \text{cm/s}^2 \]

V.Imp. 3) A Co-ordinate system \( S^* \) is rotating with respect to an inertial co-ordinate system \( S \). The two systems have the same origin and the angular velocity of \( S^* \) relative to \( S \) is given by
\[ \ddot{\omega} = 2\dot{i} + t^2 \dot{j} + tk \]

The position vector of a particle in the system \( S \) is given by \( \vec{r} = \dot{t} \dot{i} + \dot{j} + t^2 \dot{k} \)

Find
(1) The centrifugal acceleration of the particle in \( S^* \) at time \( t = 1 \text{sec.} \)
(2) Find the corilolis acceleration of a particle in \( S^* \) at \( t = 1 \text{sec.} \)

All quantities are in S.I. system of units

(1) Centrifugal acceleration
\[ \ddot{\omega} \times (\ddot{\omega} \times \vec{r}) \]
\[ \ddot{\omega} \times \vec{r} = \begin{vmatrix} \dot{i} & \dot{j} & \dot{k} \\ 2 & t^2 & t \\ t & 1 & t^2 \end{vmatrix} \]
\[ = \dot{i}[t^4 - t] - \dot{j}[2t^2 - t^3] + \dot{k}[2 - t^4] \]

At \( t = 1 \text{ sec.} \)
\[ = \dot{i} - \dot{j} + \dot{k} \]

Centrifugal acceleration
\[ \ddot{\omega} \times (\ddot{\omega} \times \vec{r}) = \begin{vmatrix} i & j & k \\ 2 & t^2 & t \\ 0 & -1 & t \end{vmatrix} \]
\[ = \dot{i}[1 + 1] - \dot{j}[2 - 0] + \dot{k}[-2 - 0] \]
\[ = 2\dot{i} - 2\dot{j} - 2\dot{k} \]
\[ = -2\dot{i} + \dot{j} - \dot{k} \]

\[ |\ddot{\omega} \times (\ddot{\omega} \times \vec{r})| = \sqrt{(2^2 + (-2)^2 + (-2)^2)} \neq \sqrt{12} = 2\sqrt{3} \text{cm/s}^2 \]

2) Corilolis Acceleration
\[ 2 \ddot{\omega} \times \frac{d^2 \vec{r}}{dt^2} \]
\[ = -4\dot{i} + 6\dot{j} + 2\dot{k} \]
\[ \vec{r} = \dot{t} \dot{i} + \dot{j} + t^2 \dot{k} \]
\[
\frac{d^2 \vec{r}}{dt^2} = \frac{d \vec{r}}{dt} \times \vec{\omega} + \vec{\omega} \times \frac{d \vec{r}}{dt}
\]

\[
\frac{d^2 \vec{r}}{dt^2} = \frac{d \vec{r}}{dt} - \vec{\omega} \times \frac{d \vec{r}}{dt}
\]

\[
\frac{d^2 \vec{r}}{dt^2} = \frac{d}{dt} \left[ \dot{\vec{r}} + \vec{j} + \vec{i} \cdot \vec{k} \right] = \dot{\vec{i}} + 2t \vec{k}
\]

\[
\frac{d^2 \vec{r}}{dt^2} = \left( \dot{\vec{i}} + 2\vec{k} \right) - \left( \vec{j} + \vec{k} \right) = \dot{\vec{i}} + \vec{j} + (2t-1) \vec{k}
\]

\[
\frac{d^2 \vec{r}}{dt^2} = \dot{\vec{i}} + \vec{j} + (2t-1) \vec{k}
\]

\[
\vec{\omega} \times \frac{d^2 \vec{r}}{dt} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
2 & 1 & 1 \\
1 & 1 & +1 \\
\end{vmatrix}
\]

\[
= \dot{\vec{j}} + \vec{k}
\]

Coriolis force
\[
= 2\vec{\omega} \times \frac{d^2 \vec{r}}{dt}
\]

\[
= -2 \vec{j} + 2 \vec{k}
\]

**Foucault’s Pendulum:**

Consider a simple pendulum with a heavy bob and infinitely long string. Equation of motion of this single particle with respect to earth is given by

\[
m \frac{d^2 \vec{r}}{dt^2} = \vec{T} + mg - 2m\vec{\omega} \times \frac{d^2 \vec{r}}{dt^2}
\]

\(T\) is the tension in the string.

Tension and \(g_{eff}\) are always in vertical plane but the horizontal component of Coriolis force \(-2m\vec{\omega} \times \frac{d^2 \vec{r}}{dt^2}\) is non-zero even though very small. There is no other force is horizontal direction. Therefore, its effect is always seen as slow rotation of plane of oscillation of the simple pendulum.

In order to find the angular velocity of the plane of oscillation, we introduce new rotating co-ordinate system \(S'\) with respect to the earth such that \(S'\) rotates with constant angular velocity \((\Omega \vec{k})\) about vertical axis passing through point of support. The angular velocity \(\Omega\) is so adjusted that all the forces acting on the bob are in the vertical plane. Therefore, plane of oscillation in \(S'\) will remain fixed. In this case then the angular velocity of plane of oscillation with respect to earth will be the same as the angular velocity of \(S'\) with respect to earth.
We have,
\[
\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{r}}{dt} + \mathbf{\Omega} \times \mathbf{r}
\]
\[
\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}}{dt^2} + 2\Omega \hat{k} \times \left( \frac{d\mathbf{r}}{dt} \right) + \mathbf{\Omega} \times \left( \mathbf{\Omega} \times \mathbf{r} \right)
\]

Third term is zero since $S'$ frame is rotating with constant angular velocity.

Substituting,
\[
\frac{m d^2\mathbf{r}}{dt^2} + 2m\mathbf{\Omega} \times \frac{d\mathbf{r}}{dt} + m\mathbf{\Omega}^2 \hat{k} \times \left( \mathbf{\hat{k}} \times \mathbf{r} \right) = \tilde{T} + mg_{eff} - 2m\mathbf{\hat{\omega}} \times \left( \frac{d\mathbf{r}}{dt} + \mathbf{\Omega} \times \mathbf{r} \right)
\]
\[
\frac{m d^2\mathbf{r}}{dt^2} = \tilde{T} + mg_{eff} - 2m\mathbf{\hat{\omega}} \times \frac{d\mathbf{r}}{dt} - m\mathbf{\Omega}^2 \hat{k} \times \left( \mathbf{\hat{k}} \times \mathbf{r} \right) - 2m\mathbf{\hat{\omega}} \times \frac{d\mathbf{r}}{dt} - 2m\mathbf{\Omega} \mathbf{\hat{\omega}} \times \left( \mathbf{\hat{k}} \times \mathbf{r} \right)
\]
\[
= \tilde{T} + mg_{eff} - 2m\mathbf{\hat{\omega}} \times \frac{d\mathbf{r}}{dt} - m\mathbf{\Omega}^2 \left[ \mathbf{\hat{k}} \left( \mathbf{\hat{k}} \cdot \mathbf{r} \right) - \mathbf{\hat{r}} \left( \mathbf{\hat{k}} \cdot \mathbf{r} \right) \right] - 2m\mathbf{\hat{\omega}} \left( \mathbf{\hat{k}} \cdot \mathbf{r} \right) - \mathbf{\hat{r}} \left( \mathbf{\hat{\omega}} \cdot \mathbf{r} \right)
\]

In the above equation $\tilde{T}$ and $mg_{eff}$, last two terms are also is the vertical plane as $\hat{k}$ is in vertical plane and position vector $\mathbf{r}$ is in the vertical plane. For small amplitudes $\frac{d\mathbf{r}}{dt}$ is in the horizontal plane. If the bracketed term $(\mathbf{\Omega} \times \mathbf{r})$ is also in the horizontal plane, then the cross product of this term with $\frac{d\mathbf{r}}{dt}$ will be in the vertical direction.

Thus, we have $(\mathbf{\Omega} \times \mathbf{r})$ in the horizontal plane. This is possible if the dot product of this term with $\mathbf{\hat{k}}$ is zero.

\[
(\mathbf{\Omega} \times \mathbf{r}) \cdot \mathbf{\hat{k}} = 0
\]
\[
\mathbf{\Omega} + \mathbf{\omega} \cos \theta = 0
\]
\[
\mathbf{\Omega} = -\mathbf{\omega} \cos \theta
\]

At the North Pole, co-latitude $\theta = 0^\circ$
\[
\therefore \mathbf{\Omega} = -\mathbf{\omega} \cos \theta
\]

In other words, plane of oscillation with simple pendulum will rotate with period of 24 hours whereas at a place on equator $\theta = 90^\circ$ $\therefore \mathbf{\Omega} = 0$, therefore plane of oscillation will not rotate. Period will be infinite.

Problem:
A particle is released near the earth’s surface in Northern hemisphere. Obtain the expression for eastward deflection of the particle.

Solution:
[Diagram]

We select $Z^*$ along the vertical direction and $Y^*$ is the horizontal plane pointing North. Therefore $X^*$ will be towards east. $\mathbf{\omega}$ has components only along $Y^*$ and $Z^*$ direction ( $\mathbf{\omega}$ lies in $Y^*Z^*$ plane)

\[
\mathbf{\omega} = (0, \omega \sin \theta, \omega \cos \theta)
\]

A particle of mass is released along $Z^*$ axis near the surface of the earth. Approximately, it has velocity along vertical direction only. But due to coriolis force, the particle is deflected along the $X^*$ i.e. along eastward direction.

Coriolis force $= -2m\mathbf{\hat{\omega}} \times \frac{d^2\mathbf{r}}{dt^2}$
\[
\begin{align*}
\mathbf{\omega} \times \left[ \frac{d^2 \mathbf{z}^*}{dt^2} \right] &= -2m(\omega \sin \theta \mathbf{\hat{y}} + \omega \cos \theta \mathbf{\hat{k}}) \\
\ &= -2m \omega \sin \theta \left( \frac{d^2 \mathbf{z}^*}{dt^2} \right) \mathbf{\hat{k}}
\end{align*}
\]

For particle falling down \( \frac{d^2 \mathbf{z}^*}{dt^2} = -g \)

Integrating, \( \frac{dz^*}{dt} = -gt + c \)

But at \( t=0 \) \( \frac{dz^*}{dt} = 0 \Rightarrow c = 0 \)

\[ \therefore \frac{dz^*}{dt} = -gt \]

Coriolis force = \( 2m \omega \sin \theta (g_{eff} \theta) \mathbf{\hat{i}} \)

Thus net Coriolis force is towards east and therefore particle is deflected eastward.

By Newton’s Laws of motion
\[ \frac{d^2 x^*}{dt^2} = 2 \omega \sin \theta g_{eff} t \]

Integrally \( \frac{dx^*}{dt} = 2 \omega \sin \theta g_{eff} \frac{t^2}{2} + c \)

At \( t=0 \) \( \frac{dx^*}{dt} = 0 \Rightarrow c = 0 \)

\[ \therefore \frac{dx^*}{dt} = \omega g_{eff} \frac{t^3}{3} \sin \theta + c \]

Integrals
\[ x^* = \omega g_{eff} \frac{t^3}{3} \sin \theta + c \]

At \( t=0 ; x^*=0 \Rightarrow c=0 \)

\[ \therefore x^* = \omega g_{eff} \frac{t^3}{3} \sin \theta \]

Where, \( t \) is the time taken by the particle to reach the ground (surface of earth).
It is called time of fall.

To find \( t \)

We have \( \frac{dz^*}{dt} = -g_{eff} t \)

Integrally \( z^* = -\frac{1}{2} g_{eff} t^2 + c \)

At \( t=0 \), \( z^* = h \) height through which the particle is released \( \therefore c = h \)

\[ z^* = -\frac{1}{2} g_{eff} t^2 + h \]

\[ \therefore z^* = h - \frac{1}{2} g_{eff} t^2 \]

At \( t \) = time of fall when the particle reaches the ground \( z^*=0 \)

\[ \therefore 0 = h - \frac{1}{2} g_{eff} t^2 \]

\[ \therefore t = \left( \frac{2h}{g_{eff}} \right)^{\frac{1}{2}} \]
Substituting this is expresser of $x^*$ we get,

$$x^* = \frac{1}{3} \omega \left( \frac{2h}{\gamma_{g,e}} \right)^{\frac{3}{2}} g_{e,g} \sin \theta$$

If $\lambda^*$ is latitude the co-latitude $\theta = 90 - \lambda$

$$x^* = \frac{1}{3} \omega \left( \frac{2h}{\gamma_{g,e}} \right)^{\frac{3}{2}} g_{e,g} \cos \lambda$$

Larmor’s Theorem:

1) There is similarity between Coriolis force and magnetic force on a moving charge particle.

$$q \frac{d\vec{r}}{dt} \times \vec{B} \text{ and } 2m \frac{d\vec{r}}{dt} \times \omega$$

Both these forces depend on velocity of the particle. Therefore, the effect of magnetic field can be simulated by introducing rotating frame.

2) Lorentz magnetic force on a moving charge particle is equivalent to rotating co-ordinate system. This equivalence is called Larmor’s Theorem.

Statement:
Consider a system of charged particles having the same specific charge (i.e. $q/m$ same for all the particle). It is acted upon by their mutual forces and by a common central force towards a common centre. In addition to these forces, the system is subject to weak uniform magnetic field $\vec{B}$. Its possible motions will be the same as the motions it could perform in the absence of magnetic field superimposed upon slow processional motion of the entire system about the centre of force with angular velocity given by $\vec{\omega} = -\frac{q}{2m} \vec{B}$

This is known as Larmor’s Theorem.

Proof:
For simplicity, consider a system of particles having the same charge and same mass for the entire particle. The equation of motion for the system of particles is

$$m \frac{d^2\vec{r}_j}{dt^2} = \vec{F}_j^i + \vec{F}_j^c \quad j=1,2,\ldots,N \quad (1)$$

Where $\vec{F}_j^i$ is the internal mutual interaction force acting on the $j$th particle due to other particles in the system and $\vec{F}_j^c$ is a common central force acting towards common centre.

When magnetic field is applied, the equation of motion becomes,

$$m \frac{d^2\vec{r}_j}{dt^2} = \vec{F}_j^i + \vec{F}_j^c + \frac{q}{m} \frac{d\vec{F}_j}{dt} \times \vec{B} \quad (2)$$

[Remark (3) we introduce a rotating frame $S^*$ rotating with constant angular velocity such that the equation number 2 reduces to

$$m \frac{d^2\vec{r}_j}{dt^2} = \vec{F}_j^i + \vec{F}_j^c \quad (3)$$

Thus, the possible motions of the particles as described by the equation 2 and 3 are the same. Thus, the possible motions in presence of magnetic field would be the same as in the absence of magnetic field but in the rotating co-ordinate system]

We have
\[
\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} + \frac{d^2 \vec{r}}{dt^2}
\]

substituting in 1

\[
m \frac{d^2 \vec{r}}{dt^2} + 2m\vec{\omega} \times \frac{d\vec{r}}{dt} + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{F}_i^j + \vec{F}_i^c + q \left( \frac{d\vec{r}}{dt} + \vec{\omega} \times \vec{r} \right) \times \vec{B}
\]

\[
\therefore \ m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_i^j + \vec{F}_i^c - 2m\vec{\omega} \times \frac{d\vec{r}}{dt} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) + q \left( \frac{d\vec{r}}{dt} + \vec{\omega} \times \vec{r} \right) \times \vec{B}
\]

\[
= \vec{F}_i^j + \vec{F}_i^c + \frac{d\vec{r}}{dt} \times (2m\vec{\omega} + q\vec{B}) - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - q\vec{B} \times (\vec{\omega} \times \vec{r})
\]

The constant angular velocity of rotating frame is so adjusted, that the bracketed term on R.H.S. becomes zero.

\[
2m\vec{\omega} + q\vec{B} = 0
\]

\[
\Rightarrow 2m\vec{\omega} = -q\vec{B}
\]

\[
\therefore \ \vec{\omega} = -\frac{q}{2m} \vec{B}
\]

\[
\therefore \ m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_i^j + \vec{F}_i^c - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - q\vec{B} \times (\vec{\omega} \times \vec{r})
\]

\[
= \vec{F}_i^j + \vec{F}_i^c - m\vec{\omega} \times \left( \left[ -\frac{q}{2m} \vec{B} \right] \times \vec{r} \right) - q\vec{B} \times \left[ -\frac{q}{2m} \vec{B} \right] \times \vec{r}
\]

\[
\therefore \ m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_i^j + \vec{F}_i^c - \frac{q^2}{4m} \vec{B} \times (\vec{B} \times \vec{r}) - \frac{q^2}{2m} \vec{B} \times (\vec{B} \times \vec{r})
\]

\[
m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_i^j + \vec{F}_i^c - \frac{q^2}{4m} \vec{B} \times (\vec{B} \times \vec{r})
\]

Since the applied magnetic field is weak

\[
\left| \vec{B} \times (\vec{B} \times \vec{r}) \right| = B^2 r_j \sin \theta \lesssim \text{small}
\]

\[
\text{B}^2 \text{ is negligible in comparison with the other forces.}
\]

Thus, we get

\[
\therefore \ m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_i^j + \vec{F}_i^c \quad (3)
\]

Hence, the Larmor’s theorem is proved.

Degree of freedom

The minimum number of independent parameters or co-ordinates required to describe the system completely is known as the degree of freedom of the system.

1) A particle free to move in free space: the degree of freedom is \textbf{three}.
2) A rigid body: the degree of freedom is \textbf{six} – three co-ordinates are required to specify the C.M. of the rigid body and three more co-ordinates are required to specify the orientation of rigid body.

The degree of freedom \( n \) of a system of \( N \) particles is given by; \( n = 3N - k \) where, \( k \) is number of constraints.

A constraint is certain restriction on the motion of particle. We consider only those constraints, which can be expressed in the forms of relation between the co-ordinates of system. Such constraints are known ‘Holonomic constraints’
Consider e.g. degree of freedom of a system consisting of 3 particles with the constraint that the distance between two particles is always constant. The degree of freedom of such a system is six.

\[ n = 3(n_{12} + n_{13} + n_{23}) - 3 = 9 - 3 \]
\[ n = 6 \]

Extending this to a rigid body consisting of very large number of particles with the same constraints that the distance between any two particles is always constant, the degree of freedom of the rigid body is therefore six \((r_{ij} = \text{constant})\)

**Generalised velocity**

Time derivative of generalised co-ordinate is called the corresponding generalised velocity. Generalised velocities are denoted by \(\{q, q, \ldots, q, \ldots\}\)

The generalised velocities of a particle freely moving in three dimension is \(\{r, \theta, \phi\}\) or \(\{r, \phi, z\}\)

**Generalised momentum**

In Cartesian co-ordinates, the kinetic energy of a particle in 1-dimension is given by

\[ K = \frac{1}{2} m \dot{x}^2 \]

The ordinary momentum of a particle is defined as

\[ p = m \dot{x} \]

This can be expressed as the derivative of K.E. \(K\) with respect to \(\dot{x}\)

\[ \Rightarrow p = m \dot{x} = \frac{\partial K}{\partial \dot{x}} \]

On this line, generalised momentum corresponding to the generalised co-ordinate \(q\) can be defined as:

\[ p = \frac{\partial K(q, \dot{q})}{\partial \dot{q}} \]

The Cartesian co-ordinates and the generalised co-ordinates of a particle in one dimension is related by the equation:

\[ x = x[q, \dot{q}] \]
\[ q = q[x, \dot{x}] \]

e.g.

\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \theta \]

i.e.

\[ x = x[r, \theta, \phi] \]

Also, \(r = \sqrt{x^2 + y^2 + z^2}, r = r[x, y, z]\

\[ r_{12} = \text{constant}, \quad r_{23} = \text{constant}, \quad r_{13} = \text{constant} \]
\[ P = \frac{\partial K(\dot{q}, t)}{\partial \dot{q}} \]  
\[ = \frac{\partial K(x, t)}{\partial \dot{x}} \cdot \frac{\partial \dot{x}}{\partial \dot{q}} \]  
\[ P = p \frac{\partial \dot{x}}{\partial \dot{q}} \]  

But \( x = x[q(t), t] \) (explicit dependence on time)  
\[ \therefore \dot{x} = \frac{\partial x}{\partial \dot{q}} \cdot \dot{q} + \frac{\partial x}{\partial t} \]  

(*)

Differentiating with respect to \( \dot{q} \)
\[ \frac{\partial \dot{x}}{\partial \dot{q}} = \frac{\partial x}{\partial \dot{q}} \]

We always treat \( q \) and \( \dot{q} \) as independent variables

A) \[ [F(x(t), t)] \]
\[ \frac{dF}{dt} = \frac{\partial F}{\partial x} \dot{x} + \frac{\partial F}{\partial t} \]  
\[ \therefore P = p \frac{\partial \dot{x}}{\partial \dot{q}} \]

Differentiating with respect to \( \dot{t} \)
\[ \therefore P = p \frac{\partial \dot{x}}{\partial \dot{q}} + p \frac{d}{dt} \left( \frac{\partial x}{\partial \dot{q}} \right) \]

We have,
\[ \frac{d}{dt} \left( \frac{\partial x}{\partial \dot{q}} \right) = \frac{\partial}{\partial \dot{q}} \left( \frac{\partial x}{\partial \dot{q}} \right) \dot{q} + \frac{\partial}{\partial \dot{t}} \left( \frac{\partial x}{\partial \dot{q}} \right) \]  

(using *)

\[ = \frac{\partial}{\partial \dot{q}} \left( \frac{\partial x}{\partial \dot{q}} \right) \dot{q} + \frac{\partial}{\partial \dot{t}} \left( \frac{\partial x}{\partial \dot{q}} \right) \]  

\[ = \frac{\partial}{\partial \dot{q}} \left( \frac{\partial x}{\partial \dot{q}} \right) \dot{q} \]  

(A)

\[ = \frac{\partial}{\partial \dot{t}} \left( \frac{\partial x}{\partial \dot{q}} \right) \]  

\[ \therefore P = p \frac{\partial \dot{x}}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}} \left( \frac{\partial x}{\partial \dot{q}} \right) \]  

Using Newton’s 2nd law of motion (rate of change of momentum of a particle is directly proportional to impressed force)

\[ \dot{p} = F \]
\[ \therefore \dot{p} = F \frac{\partial x}{\partial \dot{q}} + p \frac{\partial \dot{x}}{\partial \dot{q}} \]
\[ = F \frac{\partial x}{\partial \dot{q}} + \left( \frac{\partial K}{\partial \dot{x}} \right) \frac{\partial \dot{x}}{\partial \dot{q}} \]

(Composite rule)
Generalized force in 1-dimension is defined as $F = \frac{\partial x}{\partial q}$

[Conservative force field: If the work done by the force is independent of path followed by the particle then the force is said to be conservative force.]

If the force is conservative then it can always be expressed as gradient of a scalar function. In one dimension, the one can write

$$P = -\frac{\partial V}{\partial q} + \frac{\partial K}{\partial q} + Q'$$

Where $Q'$ is non-conservative generalized force. It cannot be expressed as gradient of a scalar potential. [e.g. Friction is non-conservative force]

$$d(P) = \frac{\partial}{\partial q}(K - V) = Q'$$

Generally in most of the cases potential is independent of velocities. It depends only on co-ordinates. In that case, Consider,

$$\frac{\partial}{\partial q}(K - V) = \frac{\partial K}{\partial q} - \frac{\partial V}{\partial q}$$

$$= 0$$

$$= \frac{\partial K}{\partial q}$$

$$d\left[\frac{\partial}{\partial q}(K - V)\right] - \frac{\partial}{\partial q}(K - V) = Q'$$

Let $L = K - V$

$\therefore L$ is called as Lagrangian function defined as the difference of K.E. and P.E.

$$d\left[\frac{\partial L}{\partial q}\right] - \frac{\partial L}{\partial q} = Q'$$

It is the Langrangian equation in 1-dimension. In most of the physical situations, the particle does not experience any non-conservative forces. Therefore in absence of non-conservative forces $Q' = 0$, the Lagrange's equation in one dimension becomes

$$d\left(\frac{\partial L}{\partial q}\right) - \frac{\partial L}{\partial q} = 0$$

N particle no constraints

$$x_i = x_i[q_i(t),q_{i1}(t),...,q_{ik}(t),t]$$
$$y_i = y_i[q_i(t),q_{i1}(t),...,q_{ik}(t),t]$$
$$z_i = z_i[q_i(t),q_{i1}(t),...,q_{ik}(t),t]$$

$k = 1,2,3...3N$
\[ q_i = q_i[x_i(t), y_i(t), z_i(t), t] \]
\[ \dot{x}_i = \sum \frac{\partial x_i}{\partial q_i} \dot{q}_i + \frac{\partial x_i}{\partial t} \]
\[
K = \frac{1}{2} \sum_i m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) 
\]
\[ p_{x_i} = \frac{\partial K}{\partial \dot{x}_i} \]
\[ p_{x_i} = \frac{\partial K}{\partial \dot{q}_i} \]
\[ = \sum \frac{\partial K}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial q_i} + \sum \frac{\partial K}{\partial \dot{y}_i} \frac{\partial \dot{y}_i}{\partial q_i} \]

Lagrange's equation in several dimensions

Generalized \( q_1, q_2 \ldots q_\text{N} = \{q_k, k = 1,2,3N \}

Cartesian \( x_1, y_1, z_1, \ldots x_N, y_N, z_N; i = 1,2,3N \)

\[ K = \frac{1}{2} \sum_i m_i \{\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\} \]
\[ \therefore \frac{\partial K}{\partial \dot{x}_i} = \frac{1}{2} m_2 \dot{x}_i = m_i x_i = p_{x_i} \]

We have,
\[ x_i = x_i[q_i(t), t] \]

Similarly for \( y_i \) and \( z_i \)
\[ q_i = q_i[x_i(t), y_i(t), z_i(t), t] \]
\[ K = K[x_i, \dot{x}_i, \dot{z}_i, t] \]
\[ K = K[q_i, \dot{q}_i, t] \]

Definition
\[ p_k = \frac{\partial K}{\partial \dot{q}_k}[x_i, \dot{y}_i, \dot{z}_i] \]
\[ p_k = \sum \left[ \frac{\partial K}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial x_i} + \frac{\partial K}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial y_i} + \frac{\partial K}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial z_i} \right] \]

Use \[ \frac{\partial x_i}{\partial q_k} = \frac{\partial x_i}{\partial q_k} \]
(Similarly for \( y_i \) and \( z_i \))
\[ p_k = \sum \left[ p_{x_i} \frac{\partial x_i}{\partial q_k} + p_{y_i} \frac{\partial y_i}{\partial q_k} + p_{z_i} \frac{\partial z_i}{\partial q_k} \right] \]

Differentiating with respect to \( t \)
\[ \dot{p}_k = \sum p_{x_i} \left( \frac{\partial x_i}{\partial q_k} \right) + p_{y_i} \frac{d}{dt} \left( \frac{\partial x_i}{\partial q_k} \right) + p_{y_i} \frac{d}{dt} \left( \frac{\partial y_i}{\partial q_k} \right) + p_{z_i} \frac{d}{dt} \left( \frac{\partial z_i}{\partial q_k} \right) + p_{y_i} \frac{d}{dt} \left( \frac{\partial y_i}{\partial q_k} \right) + p_{z_i} \frac{d}{dt} \left( \frac{\partial z_i}{\partial q_k} \right) \]
Using Newton’s law of motion

\[ m \ddot{x}_i = F_n = \frac{d}{dt}(m, \dot{x}_i) = p_i \]

\[ \dot{p}_k = \sum_i \left[ F_{ix} \left( \frac{\partial \chi_i}{\partial q_k} \right) + F_{iy} \left( \frac{\partial \gamma_i}{\partial q_k} \right) + F_{iz} \left( \frac{\partial \zeta_i}{\partial q_k} \right) \right] + \sum_i \left[ p_{ix} \frac{d}{dt} \left( \frac{\partial \chi_i}{\partial q_k} \right) + p_{iy} \frac{d}{dt} \left( \frac{\partial \gamma_i}{\partial q_k} \right) + p_{iz} \frac{d}{dt} \left( \frac{\partial \zeta_i}{\partial q_k} \right) \right] \]

Use

\[ \frac{d}{dt} \left( \frac{\partial \chi_i}{\partial q_k} \right) = \frac{\partial \chi_i}{\partial q_k} \]

\[ \dot{p}_k = \sum_i \left[ F_{ix} \left( \frac{\partial \chi_i}{\partial q_k} \right) + F_{iy} \left( \frac{\partial \gamma_i}{\partial q_k} \right) + F_{iz} \left( \frac{\partial \zeta_i}{\partial q_k} \right) \right] + \sum_i \left[ p_{ix} \frac{\partial \chi_i}{\partial q_k} + p_{iy} \frac{\partial \gamma_i}{\partial q_k} + p_{iz} \frac{\partial \zeta_i}{\partial q_k} \right] \]

\[ \dot{p}_k = \sum_i \left[ F_{ix} \left( \frac{\partial \chi_i}{\partial q_k} \right) + F_{iy} \left( \frac{\partial \gamma_i}{\partial q_k} \right) + F_{iz} \left( \frac{\partial \zeta_i}{\partial q_k} \right) \right] + \frac{\partial K}{\partial q_k} \]

Recall

\[ K = K[q_i(q_k, \dot{q}_k, t), \dot{q}_i(q_k, \dot{q}_k, t)] \]

\[ \frac{\partial K}{\partial q_k} = \sum_i \left[ \left( \frac{\partial K}{\partial \chi_i} \right) \left( \frac{\partial \chi_i}{\partial q_k} \right) + \left( \frac{\partial K}{\partial \gamma_i} \right) \left( \frac{\partial \gamma_i}{\partial q_k} \right) + \left( \frac{\partial K}{\partial \zeta_i} \right) \left( \frac{\partial \zeta_i}{\partial q_k} \right) \right] \]

The forces acting on the \( i \)th particle can be conservative or non-conservative. Conservative forces can be expressed as negative of gradient of Scalar potential. The non-conservative part is left as it is.

We have,

\[ F_n = -\frac{\partial V}{\partial x} + F_n' \quad \text{(Similar } F_{yq} \text{ and } F_{xq} \text{)} \]

\[ \dot{p}_k = \sum_i \left[ -\left( \frac{\partial V}{\partial \chi_i} \right) \left( \frac{\partial \chi_i}{\partial q_k} \right) + \left( \frac{\partial V}{\partial \gamma_i} \right) \left( \frac{\partial \gamma_i}{\partial q_k} \right) + \left( \frac{\partial V}{\partial \zeta_i} \right) \left( \frac{\partial \zeta_i}{\partial q_k} \right) \right] + \sum_i \left[ F_{ix} \left( \frac{\partial \chi_i}{\partial q_k} \right) + F_{iy} \left( \frac{\partial \gamma_i}{\partial q_k} \right) + F_{iz} \left( \frac{\partial \zeta_i}{\partial q_k} \right) \right] + \frac{\partial K}{\partial q_k} \]

\[ = -\frac{\partial V}{\partial q_k} + Q_k' + \frac{\partial K}{\partial q_k} \]

Where \( Q_k' \) is a non-conservative generalized force corresponding to generalized co-ordinate \( q_k \)

It is defined as:

\[ Q_k' = \sum_i \left[ F_{ix} \frac{\partial \chi_i}{\partial q_k} + F_{iy} \frac{\partial \gamma_i}{\partial q_k} + F_{iz} \frac{\partial \zeta_i}{\partial q_k} \right] \]

\[ \dot{p}_k = \frac{\partial K}{\partial q_k} \frac{\partial V}{\partial q_k} + Q_k' \]

\[ = \frac{\partial}{\partial q_k} [K - V] + Q_k' \]

\[ \therefore \frac{d}{dt} [p_k] = \frac{\partial}{\partial q_k} [K - V] + Q_k' \]

\[ \frac{d}{dt} \left[ \frac{\partial K}{\partial q_k} \right] - \frac{\partial}{\partial q_k} [K - V] = Q_k' \]

Potential \( V \) is a function of only \( q_k \)
\[ V = V[q_i(\dot{q}, \ddot{q})] \]

In most of the situations potential is independent of generalized velocities

\[ \Rightarrow \frac{\partial V}{\partial q_k} = 0 \]

\[ \Rightarrow \frac{\partial K}{\partial q_k} = \frac{\partial}{\partial \dot{q}_k} [K - V] \]

\[ \therefore \frac{d}{dt} \left[ \frac{\partial}{\partial \dot{q}_k} (K - V) \right] - \frac{\partial}{\partial \dot{q}_k} (K - V) = Q'_k \]

Let \( L = K - V \Rightarrow \text{Lagrangian function} \)

\[ L = L[q_k(t), \dot{q}_k(t), t] = K - V \]

\[ \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{q}_k} \right] - \frac{\partial L}{\partial q_k} = Q'_k \quad k = 1, 2, \ldots 3N \]

If the particles do not experience any non-conservative force i.e. \( Q'_k = 0 \), then the Lagrange's equations become

\[ \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{q}_k} \right] - \frac{\partial L}{\partial q_k} = 0 \quad k = 1, 2, \ldots 3N \]

Imp.

To prove,

\[ \frac{\partial \dot{x}_k}{\partial \dot{q}_k} = \frac{\partial \dot{x}_i}{\partial q_k} \]

Proof we have,

\[ x_i = x_i[q_k(t), t] \quad k = 1, 2, \ldots 3N \]

\[ \frac{d}{dt}(x_i) = \dot{x}_i = \frac{\partial x_i}{\partial q_k} \left( \frac{d}{dt} q_k \right) + \frac{\partial x_i}{\partial \dot{q}_k} \]

\[ \dot{x}_i = \sum_k \left( \frac{\partial x_i}{\partial q_k} \right) \dot{q}_k + \left( \frac{\partial x_i}{\partial \dot{q}_k} \right) \]

Differentiating with respect to \( \dot{q}_k \)

(Generalised co-ordinates and generalised velocities are always treated as independent variables)

\[ q_k = 1, 2, \ldots 3N \]

In other words when partially differentiating with respect to \( \dot{q}_k \), \( q_k \) is kept constant.

\[ \because \left( \frac{\partial \dot{x}_k}{\partial q_k} \right) \left( \frac{d}{dt} q_k \right) \]

\[ \Rightarrow \frac{\partial \dot{x}_k}{\partial q_k} = \frac{\partial x_i}{\partial \dot{q}_k} \]

2) \( \frac{d}{dt} \left( \frac{\partial x_i}{\partial q_k} \right) = \frac{\partial \dot{x}_i}{\partial q_k} \)

We have,

\[ x_i = x_i[q_k(t), t] \]
\[ \frac{\partial x_i}{\partial q_k} = \frac{\partial x_i}{\partial q_k} [q_i(t), t] \]

Differentiating with respect to time,
\[
\frac{d}{dt} \left( \frac{\partial x_i}{\partial q_k} \right) = \sum_j \left( \frac{\partial}{\partial q_j} \frac{\partial x_i}{\partial q_k} \right) \frac{\partial x_j}{\partial t} + \frac{\partial}{\partial t} \left( \frac{\partial x_i}{\partial q_k} \right)
\]
\[
= \sum_j \left( \frac{\partial}{\partial q_j} \frac{\partial x_i}{\partial q_k} \right) \dot{q}_j + \frac{\partial}{\partial t} \left( \frac{\partial x_i}{\partial q_k} \right)
\]
\[
= \frac{\partial}{\partial q_i} \left[ \sum_j \frac{\partial x_j}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t} \right]
\]
\[
= \frac{\partial \mathbf{x}}{\partial q_i}
\]

1) Atwood’s Machine:

Obtain the Lagrange function for Atwood’s Machine and hence write the equation of motion:

(Obtain the acceleration of the system)

Constraints:

1) \( y_1 = 0 \)
2) \( z_1 = 0 \)
3) \( y_2 = 0 \)
4) \( z_2 = 0 \)
5) \( l \) is constant \( (x_1 + x_2 = l) \)

\( n = 3(2) - 5 = 1 \)

\( \therefore \) Degree of freedom of Atwood’s machine is one. Hence one generalized co-ordinate is required to describe the motion.

\( q_k = x \)

Kinetic Energy:

\( K = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left( l - \dot{x} \right)^2 \)

Potential Energy:

\( V = -m_1 g x - m_2 g (l - x) \)

\( L = K - V \)

\( = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left( l - \dot{x} \right)^2 + m_1 g x - m_2 g (l - x) \)

Lagrange’s equation:

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \]
Note:
In Lagrange’s Equation the force of constraints does not appear. Hence directly one can not find the force of constraint. In order to determine the force of constraints we assume that the constraints on motion is violated for small displacements. This way, it is possible to determine the force of constraints by writing the Lagrange’s equation corresponding to the generalised co-ordinates violating the constraints.

2) Obtain the Lagrange function for simple pendulum and write equation of motion.

Number of constraints:
1) It can oscillate only in XY plane \( \Rightarrow z = 0 \)
2) Length is constant. \( l = \sqrt{x^2 + y^2} \)
\( \therefore \) Degree of freedom = \( 3(1) - 2 = 1 \)

Generalized co-ordinate \( q_k = \theta \)
\[ \begin{align*}
\dot{x} &= l \sin \theta \\
\dot{y} &= l \cos \theta \\
\dot{\theta} &= -l \sin \theta \dot{\theta}
\end{align*} \]

Kinetic Energy:
\[ K = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \]
\[ = \frac{1}{2} m l^2 \dot{\theta}^2 (\cos^2 \theta + \sin^2 \theta) \]
\[ = \frac{1}{2} m l^2 \dot{\theta}^2 \]

Potential Energy:
\[ V = mgh \]
\[ = mg(OA-OB) \]
\[ = mg(l - l \cos \theta) \]
\[ V = mg(l - \cos \theta) \]

\[ L = K - V \]
\[ = \frac{1}{2} ml^2 \dot{\theta}^2 - mg(l - \cos \theta) \]

Lagrange’s Equation
\[
\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{\theta}} \right] - \frac{\partial L}{\partial \theta} = 0
\]
\[
\frac{\partial L}{\partial \theta} = ml^2 \dot{\theta}
\]
\[
\frac{\partial L}{\partial \dot{\theta}} = -mg \sin \theta
\]
\[
\therefore \frac{d}{dt} \left[ ml^2 \dot{\theta} \right] + mg \sin \theta = 0
\]
\[ ml^2 \ddot{\theta} + mg \sin \theta = 0 \]
\[ \ddot{\theta} + \frac{g}{l} \sin \theta = 0 \]
\[ \ddot{\theta} = -\left( \frac{g}{l} \right) \sin \theta \]

Since \( \theta \) is very small, \( \sin \theta \approx \theta \)
\[ \therefore \dot{\theta} = -\left( \frac{g}{l} \right) \theta \]

Constraints:-

A force of constraint puts some restrictions on the motion of the particle. In the usual Lagrange’s method, the constraint on the motion is taken into account and accordingly only the necessary co-ordinates are considered. By this method, it is not possible to determine the force of constraint. In order to find the force of constraints, Lagrange’s function is modified to take into account that co-ordinate which is affected by the force of constraint. By writing the Lagrange’s equation for this co-ordinate, it is possible to obtain corresponding force of constraint. This method is illustrated with the help of an example of simple pendulum.

The degree of freedom of simple pendulum, with the constraint that the distance of bob from the fixed support is always constant, is one. To determine the force of constraint we introduce extra co-ordinate ‘\( r \)’ which takes constant value \( r = l \) when constraint is imposed.
The reference point for potential is taken as the point of support.

\[ V = -mg \cos \theta \]

As force of constraint is acting on the particle there is corresponding P.E. of the particle. This potential is minimum whenever the constraint i.e. \( r = l \) is satisfied. Let the potential due to this constraint force be \( V_{\text{constraint}} \)

Thus the modified Lagrange’s function is

\[ L' = \frac{1}{2}(r^2 + r^2 \dot{\theta}^2) + mgr \cos \theta - V_{\text{constraint}} \]

The Lagrange’s equation for the radial co-ordinates is given by

\[
\frac{d}{dt} \left[ \frac{\partial L'}{\partial \dot{r}} \right] - \frac{\partial L'}{\partial r} = 0
\]

Due to the deep and narrow potential \( V_{\text{constraint}} \) only one value of \( r \) i.e. \( r = l \) (constant) is allowed.

Then the above equation reduces to

\[ -ml \dot{\theta}^2 - mg \cos \theta = -\frac{\partial V_{\text{constraint}}}{\partial r} = Q_{\text{constraint}} \]

(Similar to \( E = -\bar{V}_V \))

The direction of this force of constraint is along the string inward if the string is tight. This gives us the tension in the string. In the equilibrium state \( r = l \) (constant) implies that \( \frac{\partial L'}{\partial r} = 0 \)

In other words, from the expression for \( L' \) we get,

\[
\frac{\partial L}{\partial r} - \frac{\partial V_{\text{constraint}}}{\partial r} = 0
\]
Where \( L = \left[ \frac{1}{2} m r^2 \dot{\theta}^2 + m g r \cos \theta \right]_{\text{rel}} \)

\[
Q_{\text{constraint}} = -\frac{\partial V_{\text{constraint}}}{\partial r}
\]

\[
Q_{\text{constraint}} = -\frac{\partial L}{\partial r} |_{\text{rel}}
\]

This way the force of constraint can be determined. So far we have considered Holonomic constraints acting on the system of particle. Holonomic constraint is the one that can be expressed as a relation between the co-ordinates of the system.

e.g. Rigid body – The constraint is \( r_i - r_j = \text{constant} \)
i.e. the distance between any two particles is always constant. This is relation between co-ordinates. Hence it is a Holonomic constraint.

Generalization:-

Consider a system of \( n \) particles with \( 3N \) as the total number of co-ordinates. Let there be \( c \) number of constraints on the motion of the system. These constraints are assumed to be Holonomic i.e. they can be expressed as a relation between the co-ordinates. Out of \( 3N \) co-ordinate first \( c \) co-ordinates are taken as dependant co-ordinate.

\[
f_j([x_j(t),t]) = 0 \quad k = 1,2,3...3N \quad j = 1,2...c
\]

In the case of simple pendulum the relation between the co-ordinates can be written as \( r - l = 0 \)

Out of \( 3N \) co-ordinates first \( c \) co-ordinate are selected as dependant co-ordinates. They can be selected as

\[
q_j = f_j([x_j(t),t]) = 0
\]
i.e. \( q = 0 \quad j = 1,2,3...c \)

These first \( c \) co-coordinate gives rise to a potential, which is called potential due to the force of constraints. This potential due to constraint depends on the first \( c \) coordinates \( V_{\text{constraint}}(q_1, q_2, ..., q_c, t) \)

This potential due to constraints has very deep and narrow minimum at \( (q_1, q_2, ..., q_c, t) = 0 \)

The Lagrange function is given by

\[
L' = K - V - V_{\text{constraint}}
\]

\[
L = L - V_{\text{constraint}}
\]

The solution of the possible motion of the system can be obtained by solving the following equations

\[
\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{q}_j} \right] - \frac{\partial L}{\partial q_j} = 0 \quad j = c+1, c+2...3N
\]

In order to determine the force of constraint then we introduce the first \( c \) co-ordinates and we get,

\[
Q_{\text{constraint}} = -\frac{\partial L}{\partial q_j} \quad j = 1,2...c
\]
Simple Pendulum with oscillating support:

Consider a simple pendulum with both of mass ‘m’ attached to one free end the other end is attached to a support which is not fixed but can oscillate.

(Diagram)

(Degree of freedom for simple pendulum: the degree of freedom is one)

But since support is not fixed, but oscillating it can oscillate is sidewise or vertically up and down. Thus it has two degrees of freedom. Hence, degree of freedom in this case is three \( N = 3N - k \)

Here \( N = 2 \), the bob and 2\text{nd} one is oscillating support.

But they can oscillate in vertical plane only

Thus, \( 1) z_{bob} = 0 \quad 2) z_s = 0 \)

Also the length of the string is constant

3) \( x_2 + y_2 = l = \text{constant} \)

.: Thus \( n=3(2) - 3 = 3 \)

Three independent co-ordinate \( x_n, y_n, \theta \)

Kinetic Energy \( K = \frac{1}{2}m(\dot{x} + \dot{x}_n)^2 + \frac{1}{2}m(\dot{y} + \dot{y}_n)^2 \)

\[ = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{x}_n + 2\dot{y}\dot{y}_n + \dot{x}_n^2 + \dot{y}_n^2) \]

\( x = l\sin \theta \quad \dot{x} = l\cos \theta \frac{d\theta}{dt} \)

\( y = l\cos \theta \quad \dot{y} = -l\sin \theta \frac{d\theta}{dt} \)

\( \therefore \quad K = \frac{1}{2}m[l^2\dot{\theta}^2 + 2l\cos \theta \dot{x}\dot{\theta} - 2l\sin \theta \dot{x}_n \dot{\theta} + \dot{x}_n^2 + \dot{y}_n^2 + mg(l\cos \theta + y_n)] \)

Potential Energy

\( V = -mg(y + y_n) \)

\( = -mg(l\cos \theta + y_n) \)

\( L = K - V \)

\( = \frac{1}{2}m[l^2\dot{\theta}^2 + 2l\cos \theta \dot{x}\dot{\theta} - 2l\sin \theta \dot{x}_n \dot{\theta} + \dot{x}_n^2 + \dot{y}_n^2 + mg(l\cos \theta + y_n)] \)

.: Lagrange’s equation

\( \frac{d}{dt}\left[ \frac{\partial L}{\partial \dot{\theta}} \right] - \frac{\partial L}{\partial \theta} = 0 \)

\( \frac{\partial L}{\partial \theta} = ml^2\ddot{\theta} + mx_1\cos \theta - my_1l\sin \theta \)

\( \frac{d}{dt}\left[ \frac{\partial L}{\partial \dot{\theta}} \right] = m\dot{\theta}^2 \dot{\theta} + mx_1\cos \theta - my_1l\sin \theta - mx_1\dot{\theta}\dot{\theta} - my_1l\dot{\theta} \cos \theta \)

\( \frac{\partial L}{\partial \dot{\theta}} = -mx_1l\sin \theta \dot{\theta} - my_1l\cos \theta \dot{\theta} - mgl\sin \theta \)

.: Lagrange equation

\( ml^2\ddot{\theta} + mx_1\cos \theta - my_1l\sin \theta - mx_1\dot{\theta}\dot{\theta} - my_1l\dot{\theta} \cos \theta + mx_1l\dot{\theta} - my_1l\cos \theta + mgl\sin \theta = 0 \)

Divide by \( ml^2 \)

\( \ddot{\theta} + \frac{2}{l}\sin \theta - \frac{\dot{y}_n}{l}\sin \theta + \frac{\dot{x}_n}{l}\cos \theta = 0 \)
\[ \ddot{\theta} + \left( \frac{g}{l} - \frac{\dot{y}}{l} \right) \sin \theta = -\frac{x}{l} \cos \theta \]

\(\dot{y}\) and \(\dot{x}\) are the accelerations of the support. Let us assume that the support oscillates along the horizontal direction and there is no motion along vertical direction.

\[ \therefore \text{Let } x_i = x_0 \cos \omega t \quad \text{(Oscillatory part)} \]
& \( y_i = 0 \) i.e. (No up and down oscillation)
\[ \Rightarrow \dot{x}_i = -x_0 \omega \sin \omega t \]
\[ \ddot{x}_i = -x_0 \omega^2 x_0 \cos \omega t \]
\[ = -\omega^2 x_0 \cos \omega t \]
\[ \ddot{y}_i = 0 \]

Equation of motion now becomes
\[ \ddot{\theta} + \frac{g}{l} \sin \theta = -\frac{\omega^2}{l} x_0 \cos \omega t \cos \theta \]

Let the amplitude be small \( \theta \to 0 \)
\[ \Rightarrow \sin \theta \approx \theta, \quad \cos \theta \approx 1 \]
& Let \( \left( \frac{g}{l} \right) = \omega_0^2 \) Natural frequency of simple Pendulum
\[ \ddot{\theta} + \omega_0^2 \theta = \left( \frac{x_0 \omega_0^2}{l} \right) \cos \omega t \]

This is an equation of forced oscillations. The pendulum performs simple harmonic oscillations under the influence of periodic force. If the frequency of oscillation of support is exactly same as the natural frequency of the pendulum; then the pendulum will oscillate with maximum amplitude.

<table>
<thead>
<tr>
<th>Constraints</th>
<th>Holonomic</th>
<th>Non – Holonomic</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) For rigid body, ( \vec{r}_i - \vec{r}_j ) = constant, ( j \neq i = 1,2...N )</td>
<td>( 1) ) An insect kept in a rectangular box. It cannot go out. ( x \leq a, y \leq b, z \leq c )</td>
<td>( 1) ) An insect kept in a rectangular box. It cannot go out. ( x \leq a, y \leq b, z \leq c )</td>
</tr>
<tr>
<td>2) For Simple pendulum ( r = l ), i.e. ( r - l = 0 )</td>
<td>( 2) ) An insect kept in a sphere of radius ( R ) ( x^2 + y^2 + z^2 \leq R^2 )</td>
<td>( 2) ) An insect kept in a sphere of radius ( R ) ( x^2 + y^2 + z^2 \leq R^2 )</td>
</tr>
<tr>
<td>3) A wheel rolling</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rhenomic Time independent</td>
<td>Scaleronomic Time dependent</td>
<td>Rhenomic Time independent</td>
</tr>
<tr>
<td>( \vec{r}_i - \vec{r}_j ) = constant</td>
<td>An insect crawling on a balloon and we are passing air inside the balloon.</td>
<td></td>
</tr>
</tbody>
</table>

Problem:
1) Consider a particle of mass ‘m’ moving in a plane and a central force acts on it. Obtain the Lagrange function and hence write Lagrange’s equation.

(Diagram)

Origin of the co-ordinate system is taken as the centre of force. Degree of freedom is 2 since a particle is confined to a plane.

\[ : x = r \cos \theta \Rightarrow \dot{x} = \dot{r} \cos \theta - r \sin \theta \dot{\theta} \]
\[ y = r \sin \theta \Rightarrow \dot{y} = \dot{r} \sin \theta + r \cos \theta \dot{\theta} \]

**Kinetic energy**

\[
K = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \left[ (\dot{r} \cos \theta - r \sin \theta \dot{\theta})^2 + (\dot{r} \sin \theta + r \cos \theta \dot{\theta})^2 \right] = \frac{1}{2} m [\dot{r}^2 + r^2 \dot{\theta}^2]
\]

Central force, by definition is given by

\[
F(r) = F(r) \dot{r} = -\frac{k}{r^2} \dot{r} \quad \text{(Inverse square Law)}
\]

(-ve sign shows the force is inwards)

\[
: V(r) = -\int_{0}^{\infty} F(r) dr
\]

The potential energy function does not involve the co-ordinate \( \theta \). It depends only on the distance of the particle from the centre i.e. the co-ordinate ‘\( r \)’.

\[ L = K - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r) \]

The Lagrange’s equation for two co-ordinates \( r \) and \( \theta \) are as follows

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0
\]

\[
\frac{d}{dt} \left[ mr \right] - m \dot{r} \dot{\theta}^2 + \frac{\partial V(r)}{\partial r} = 0
\]

\[
m \ddot{r} - m \dot{r} \dot{\theta}^2 + \frac{\partial V(r)}{\partial r} = 0
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - 0 = 0
\]

\[
\frac{\partial L}{\partial \theta} = \text{constant}
\]

\[
m \dot{r}^2 = \text{constant}
\]

If Lagrange does not involve particular co-ordinate then the co-ordinate is said to be cyclic or ignorable co-ordinate. The corresponding generalized momentum is always conserved. Thus, cycle co-ordinate always leads to some kind of conservation principle.

If the co-ordinate time is absent in Lagrange’s function (i.e. cycle coordinate), then the corresponding conjugate momentum i.e. energy is always conserved.

2) Find the Lagrange function and hence equation of motion of compound pendulum.

(Diagram)

Degree of freedom: \( 3N - k \)

Constraints (k)

1. \( z_1 = 0 \)
2. \( z_2 = 0 \)
3. \( l_1 = \text{constant} \)
4. \( l_2 = \text{constant} \)

\[ n = 3(2) - 4 = 2. \]

\[ x_i = l_i \sin \theta_i \]
\[ \dot{x}_i = l_i \cos \theta_i \dot{\theta}_i \]
\[ y_i = l_i \cos \theta_i \]
\[ \dot{y}_i = -l_i \sin \theta_i \dot{\theta}_i \]

\[ x_2 = l_1 \sin \theta_1 + l_2 \sin \theta_2 \]
\[ \dot{x}_2 = l_1 \cos \theta_1 \dot{\theta}_1 + l_2 \cos \theta_2 \dot{\theta}_2 \]
\[ y_2 = l_1 \cos \theta_1 + l_2 \cos \theta_2 \]
\[ \dot{y}_2 = -l_1 \sin \theta_1 \dot{\theta}_1 - l_2 \sin \theta_2 \dot{\theta}_2 \]

\[ K = \frac{1}{2} m_1 \left[ \dot{x}_1^2 + \dot{y}_1^2 \right] + \frac{1}{2} m_2 \left[ \dot{x}_2^2 + \dot{y}_2^2 \right] \]
\[ = \frac{1}{2} m_1 \left[ \dot{x}_1^2 + \dot{y}_1^2 \right] + \frac{1}{2} m_2 \left[ \dot{x}_2^2 + \dot{y}_2^2 \right] + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \left( \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \right) \]
\[ = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_2 \dot{y}_2^2 + m_1 l_1 \dot{\theta}_1 \dot{\theta}_2 \cos \theta_2 - m_2 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos \theta_1 \theta_2 \]

\[ V = -(m_1 g y_1 + m_2 g y_2) \]
\[ = -m_1 g l_1 \cos \theta_1 - m_2 g l_1 \cos \theta_1 - m_2 g l_2 \cos \theta_2 \]

\[ L = K - V \]
\[ = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_2 \dot{y}_2^2 + m_1 l_1 \dot{\theta}_1 \dot{\theta}_2 \cos \theta_2 - m_2 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos \theta_2 \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i} = 0 \]

\[ \frac{\partial L}{\partial \theta_i} = m_1 l_1^2 \ddot{\theta}_i + m_1 l_1 \dot{\theta}_1 \dot{\theta}_i + m_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_i} \right) = m_1 l_1^2 \ddot{\theta}_i + m_1 l_1 \dot{\theta}_1 \dot{\theta}_i + m_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \left[ \dot{\theta}_1 \cos (\theta_1 - \theta_2) - \dot{\theta}_1 \sin (\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2) \right] \]

\[ \frac{\partial L}{\partial \dot{\theta}_1} = -m_1 g l_1 \sin \theta_1 - m_2 g l_1 \sin \theta_1 - m_2 l_1 \dot{\theta}_1 \dot{\theta}_2 \sin (\theta_1 - \theta_2) \]
\[ - m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 \dot{\theta}_1 \dot{\theta}_1 + m_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) - m_1 l_1 \dot{\theta}_1 \dot{\theta}_2 \sin (\theta_1 - \theta_2) + m_2 l_1 \dot{\theta}_1 \dot{\theta}_2 \sin (\theta_1 - \theta_2) = 0 \]
\[ m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 \dot{\theta}_1 \dot{\theta}_1 + m_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) + m_1 l_1 \dot{\theta}_1 \dot{\theta}_2 \sin (\theta_1 - \theta_2) + m_2 l_1 \dot{\theta}_1 \dot{\theta}_2 \sin (\theta_1 - \theta_2) = 0 \]
\[ \ddot{\theta}_1 \left( m_1 + m_2 \right) + g l_1 \sin \theta_1 \left( m_1 + m_2 \right) + m_2 l_1 \dot{\theta}_2 \dot{\theta}_2 \cos (\theta_1 - \theta_2) + \dot{\theta}_1^2 \sin (\theta_1 - \theta_2) = 0 \]

Special cases of compound Pendulum
Take \( m_1 = m_2 = m \) and \( l_1 = l_2 = l \)
Consider small amplitudes i.e. \( \dot{\theta}_1 \& \dot{\theta}_2 \) are very small. The equation are
\[2m_1^2 \ddot{\theta}_1 + m_1^2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_1^2 \ddot{\theta}_2 \sin(\theta_1 - \theta_2) + 2mgl \sin \theta_1 = 0\]

\(\theta_1\) and \(\theta_2\) are very small, \(\theta_1 - \theta_2\) is still small i.e. \(\theta_1 - \theta_2 \rightarrow 0\)

\(\therefore \sin(\theta_1 - \theta_2) \approx 1\), \(\therefore 2m^2 \ddot{\theta}_1 + m^2 \ddot{\theta}_2 + 0 + 2mgl \theta_1 = 0\)

\[\ddot{\theta}_1 + \ddot{\theta}_2 + \frac{g}{l} \theta_1 = 0\]

Also,

\[\frac{\partial L}{\partial \dot{\theta}_2} = m_1 l_2^2 \ddot{\theta}_2 + m_1 l_1 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\]

\[\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) = m_1 l_2^2 \dddot{\theta}_2 + m_1 l_1 \ddot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) - m_1 l_2 \dot{\theta}_2 \dddot{\theta}_1 \sin(\theta_1 - \theta_2)\]

\[= m_1 l_2^2 \dddot{\theta}_2 + m_1 l_1 \ddot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) - m_1 l_2 \dot{\theta}_2 \dddot{\theta}_1 \sin(\theta_1 - \theta_2) + m_1 l_2 \dot{\theta}_1 \dddot{\theta}_2 \sin(\theta_1 - \theta_2)\]

\[\frac{\partial L}{\partial \theta_2} = m_1 l_1 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_1 g l_2 \sin \theta_2\]

\[\frac{\partial L}{\partial \theta_2} = m_1 l_1 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_1 g l_2 \sin \theta_2\]

Lagrange’s equation

\[\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = 0\]

\[m_1 l_2^2 \dddot{\theta}_2 + m_1 l_1 \ddot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) - m_1 l_2 \dot{\theta}_2 \dddot{\theta}_1 \sin(\theta_1 - \theta_2)\]

\[+ m_1 l_2 \dot{\theta}_1 \dddot{\theta}_2 \sin(\theta_1 - \theta_2) - m_1 l_2 \dot{\theta}_2 \dddot{\theta}_1 \sin(\theta_1 - \theta_2) + m_1 g l_2 \sin \theta_2 = 0\]

\[m_1 l_2^2 \dddot{\theta}_2 + m_1 l_1 \ddot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) - m_1 l_2 \dot{\theta}_2 \dddot{\theta}_1 \sin(\theta_1 - \theta_2)\]

\[+ m_1 l_2 \dot{\theta}_1 \dddot{\theta}_2 \sin(\theta_1 - \theta_2) - m_1 l_2 \dot{\theta}_2 \dddot{\theta}_1 \sin(\theta_1 - \theta_2) + m_1 g l_2 \sin \theta_2 = 0\]

Special Case:

\[m_1 = m_2 = m, \quad l_1 = l_2 = l\]

\(\theta_1\) and \(\theta_2\) very small

\(\therefore \sin \theta_1 \approx \theta_1\), \(\therefore (\theta_1 - \theta_2) \rightarrow 0\)

\(\sin(\theta_1 - \theta_2) \rightarrow 0, \quad \cos(\theta_1 - \theta_2) \rightarrow 1\)

\[m^2 \dddot{\theta}_1 + m^2 \dddot{\theta}_2 + mg l \theta_2 = 0\]

\[\ddot{\theta}_2 + \ddot{\theta}_1 + \frac{g}{l} \theta_1 = 0\]

**Non –Linear Mechanics (Chaos)**

In Newtonian Mechanics, the laws are predictable and the equations of motions are linear. The solutions of the equations are not very sensitive to the initial conditions. If there are certain errors in specifying the initial uncertainties in the solution, there are corresponding predictable uncertainties in the solution.

There are systems, which are governed by non-linear equations, and for certain values of parameters of the equation, the solutions are quite unpredictable. In other words, the solutions are very sensitive to the initial
conditions. Small change in initial conditions makes very large change in solutions thereby making the output quite unpredictable.

The essential difference between a chaotic system and a non-chaotic one is the degree of predictability of the motion given the initial conditions to some level of accuracy.

Example: Consider two simple harmonic oscillators be driven by some sinusoidal periodic force. Let the two oscillators start with different initial conditions but after some time when transients are over both the oscillators oscillate with same frequency as that of the driver's frequency. This is the example of non-chaotic system.

The well-known example of chaotic system is the weather system.

Note: Even though Brownian motion of gas molecule is random motion, it is not a chaotic motion. This is because the motion of the gas molecule becomes unpredictable as initial conditions of the gas molecules are not known. If somebody provides the initial conditions, then the trajectory of a gas molecule can be predicted very well.

Anharmonic Oscillator: The best example of chaotic system for a certain parametric values is the Anharmonic oscillator. The simple pendulum is an approximation of an Anharmonic oscillator. In this case, the amplitude of oscillations was assumed to be very small.

Consider an Anharmonic oscillator of mass 'm' and subjected to a force $F(x) = -k(x + \alpha x^3)$

The P.E. of the oscillator is

$$V(x) = - \int_0^x F(x) \, dx$$

$$= \int_0^x k(x + \alpha x^3) \, dx$$

$$= \frac{1}{2} k x^2 + \frac{\alpha k x^4}{4}$$

$$= k \left( \frac{x^2}{2} + \frac{\alpha x^4}{4} \right)$$

Following plots give the variation of P.E. $V(x)$ as a function of $x$ for different values of $k$ and $\alpha$. For simplicity, $k$ is taken as ±1 and similarly $\alpha$ is taken as ±1.

If $\alpha = 0$ and $k$ is positive then the particle performs usual S.H.O. but if $\alpha$ is positive then the particle tries to come back to its mean position quickly as if the restoring force has increased. It is called a hard spring. If $\alpha$ is negative ($\alpha < 0$) then the spring is called soft spring.
In the above diagrams plot (a) corresponds to a single equilibrium position at \( x = 0 \). It is stable equilibrium.

Plot (b) also has only one equilibrium position at \( x = 0 \). But it is possible only for small values of energy.

Plot (c) has equilibrium at \( x = 0 \) but it is unstable.

Plot (D) has three equilibrium positions. Two are stable and one at \( x = 0 \) is unstable.

At equilibrium position
\[
f(x) = 0
\]
\[
\Rightarrow -k(x + \alpha^2) = 0
\]
\[
\Rightarrow -k(1 + \alpha^2) = 0
\]
\[
\Rightarrow x = 0
\]
or
\[
1 + \alpha^2 = 0 \Rightarrow \alpha^2 = -1 \Rightarrow x^2 = \frac{1}{\alpha}
\]

But in (d) \( \alpha = -ve \)
\[
x^2 = \frac{1}{\alpha}
\]
\[
x = \pm \frac{1}{\sqrt{|\alpha|}}
\]
\[
x = + \frac{1}{\sqrt{|\alpha|}}, - \frac{1}{\sqrt{|\alpha|}}
\]

Most stable and sustained oscillatory motions are possible for the plots (a) and (d) shown above.

Consider a particle, in addition to \( F(x) \) experiences a damping force given by
\[
F_{\text{damping}} = -2m \frac{dx}{dt}, \text{ where, } \gamma - \text{coefficient of damping per unit mass of the particle}
\]

Let external periodic force be acting on the particle given by,
\[
F_{\text{driving}} = fn \cos \omega t
\]
Where \( f \) - amplitude of periodic force per unit mass

Applying the Newton’s second law of motion, we get,
\[
m \frac{d^2x}{dt^2} = F_{\text{restoring}}(x) + F_{\text{damping}} + F_{\text{driving}}
\]
\[
\therefore \frac{md^2x}{dt^2} = -k(x + \alpha^2) - 2m \frac{dx}{dt} + fn \cos \omega t
\]
\[
\ddot{x} + 2\beta \dot{x} + \left( \frac{k}{m} \right) x + \left( \frac{k^2}{m^2} \right) x^3 = f \cos \omega t
\]

But, \( \left( \frac{k}{\sqrt{m}} \right) = \omega_n = \text{natural frequency of the oscillation} \)

Let \( \frac{k\alpha}{m} = \beta \)
\[
\therefore \ddot{x} + 2\beta \dot{x} + \omega_n^2 x + \beta x^3 = f \cos \omega t
\]
This is the equation of motion of an Anharmonic Oscillator called as Duffing’s oscillator.

By selecting the scale of time properly, the natural frequency \( \omega_0 \) can be made one. In other words, the time scale is so adjusted that the periodic time of the particle is exactly \( 2\pi \). Therefore \( \omega_0 = 1 \). Similarly, adjusting scale of \( x \) properly \( \beta \) can be made \( \pm 1 \).

In this case, the equation becomes

\[
\ddot{x} + 2\gamma \dot{x} + x \pm x^3 = f \cos \omega t
\]

This is called reduced equation of motion of Duffing’s oscillator. There are only three parameters involved, damping coefficient \( \gamma \) the external periodic force frequency and amplitude.

**Quadratic Map (Logistic Map):**

Consider the population of certain species. It depends on the food supply available. If the food supply is unlimited then population goes on increasing. But if the food supply is finite then population will attain a maximum and afterwards will decrease to zero. If we normalise the population by dividing it by the maximum possible population then population is expressed as a fraction. This fractional population in \( (n+1)^{\text{th}} \) year depends on the fractional population of the previous year non-linearly. It is given by,

\[
x_{n+1} = \lambda (1 - x_n) x_n
\]

This is known as quadratic equation or the logistic map. \( x_n \) and \( x_{n+1} \) can take the values between 0 and 1. Therefore \( \lambda \) lies between 0 and 4.

(Diagram)

At maximum

\[
\frac{dx_{n+1}}{dx_n} \bigg|_{\text{max}} = 0
\]

\[
\lambda (1 - x_n) - \lambda x_n = 0
\]

\[
1 - 2x_n = 0
\]

\[
x_n = 0.5
\]

\[
\therefore x_{n+1} = \lambda (0.5)(0.5)
\]

\[
= 0.25\lambda
\]

\[
x_{n+1} = \frac{\lambda}{4}
\]

\[
\Rightarrow 0 < \lambda < 4
\]

For different values of parameter \( \lambda \), the logistic map exhibits various aspects of chaotic systems. There are three interesting ranges of \( \lambda \).

1) \( 0 < \lambda < 1 \)
2) \( 1 < \lambda < 3 \)
3) \( 3 < \lambda < 4 \).

Consider

1) \( 0 < \lambda < 1 \)

\[
x_{n+1} = \lambda x_n (1 - x_n)
\]
Let $\lambda = 0.7$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
<td>10</td>
<td>0.0043</td>
</tr>
</tbody>
</table>

Thus for $0<\lambda<1$, population attains a stable value as zero after several iterations. The starting population may be anything but finally it will reach zero. Analytically it can be shown as follows:-

$0<\lambda<1$.

Let $x_{n+1} = x_n = \bar{x}$

$\bar{x} = \lambda \bar{x}(1 - \bar{x})$

$\therefore \bar{x}[\lambda - \lambda(1 - \bar{x})] = 0$

$\Rightarrow \bar{x} = 0$

In this case, $\bar{x} = 0$ is the fixed point. This fixed point is called as attractor because the populations in the vicinity are finally attracted towards $\bar{x} = 0$.

And,

$1 - \lambda(1 - \bar{x}) = 0$

$\lambda(1 - \bar{x}) = 1$

$1 - \bar{x} = \frac{1}{\lambda}$

$\bar{x} = 1 - \frac{1}{\lambda}$

This is not possible as $0<\lambda<1 \Rightarrow \frac{1}{\lambda} > 1$. Thus, $\bar{x}$ is $-$ve which is not allowed. Hence, there is only one attractor.

Consider,

2) $1<\lambda<3$
Let $\lambda = 2.5$

$x_{n+1} = \lambda x_n (1 - x_n)$

Let $x_0 = 0.5$

<table>
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<tr>
<th>$n$</th>
<th>$x_n$</th>
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<tbody>
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<tr>
<td>11</td>
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</tr>
</tbody>
</table>

From the above analysis, it is obvious that from $1 < \lambda < 3$ there is only one fixed point $r = \left(1 - \frac{1}{\lambda}\right) = 0.6$. It is an attractor after several iterations irrespective of the starting value of $x_n$ finally, it will converge to $r = 1 - \frac{1}{\lambda}$. The other fixed point $r = 0$ is never attained, it is called ‘repeller’.

3) Consider $\lambda = 3.2$

$x_{n+1} = \lambda x_n (1 - x_n)$

Let $x_0 = 0.5$

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<tr>
<th>$n$</th>
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<td>0.7995</td>
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<tr>
<td>8</td>
<td>0.5129</td>
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</table>
Thus for $\lambda = 3.2$, there are two fixed points which are stable called as attractors. The population fluctuates between these two fixed values. The third point in between which is the intersection of straight line and parabola is repeller.

Till $\lambda = 3$, there was only one stable fixed point. As $\lambda$ increases beyond 3 there are two fixed points. This transition as the value of $\lambda$ is raised passed a critical value 3 from one fixed point to a pair of stable period–2 points is known as a bifurcation or period doubling.

In this period doubling phenomenon $x_{n+2}$ matches with $x_n$ i.e. $x_{n+2} = x_n$

We have,

$x_{n+2} = \lambda x_n (1 - x_n)$

$= \lambda \{\lambda x_n (1 - x_n) \} = \lambda \{ \lambda x_n \} (1 - x_n)$

Let $x_{n+2} = x_n = x$ say

$\therefore x = \lambda^2 x(1-x)(1-\lambda x + \lambda x^2)$

$\Rightarrow x(1 - \lambda^2(1-x)(1-\lambda x(1-x))) = 0$

$\Rightarrow x = 0 \text{ or } \left[ 1 - \lambda^2(1-x)(1-\lambda x(1-x)) \right] = 0$

It is a cubic equation in $x$. There are three roots. The roots are

$x = \left( 1 - \frac{1}{\lambda} \right) \frac{1 + \lambda}{2\lambda} \left( 1 \pm \sqrt{\frac{\lambda - 3}{\lambda + 1}} \right)$

The graph for $x_{n+2} = F(x_0)$ cuts the region line at three points. The extreme two points are the stable attractors and the middle one is repeller.

As $\lambda$ increase further, the fixed points keep on multiplying. First, it is period-2 bifurcation then period-4 bifurcation and so on. For $\lambda \geq 3.57$, it is observed that there is not a single fixed point. The population keeps on fluctuating between zero and one. It is non-repetitive and non-periodic. We say that the system exhibits a chaotic behaviour.

Such bifurcations occur faster and faster until an infinite number of bifurcations occur at $\lambda = 3.56994$ as shown in the above graph. Denoting by $\lambda(k)$ the critical value of $\lambda$ at which the bifurcation from a stable period $k$ set of points to a stable period $k+1$ set occurs it is found that

$\lim_{k \to \infty} \lambda_k - \lambda_{k+1} = 4.669201$

This number is called Feigenbaum number. This ratio turns out to be universal for any map with a quadratic maximum and in seen in a wide range of physical problems.

V.Imp.
For any mapping function \( x_{n+1} = F(x_n) \), show that a fixed point \( x \) will be stable if 
\[
|F'(x)| < 1 \quad \text{where} \quad F'(x) = \left. \frac{dF}{dx} \right|_{x=x_0}
\]

Solution

[Point \( x \) is called fixed point if \( x_{n+1} = x_n = x \) (Constant) for large \( n \)]

\[
\therefore \frac{dF}{dx} = \lambda(1-x_n) - 2x \\
= \lambda(1-2x)
\]

\[
\therefore \left. \frac{dF}{dx} \right|_{x=x_0} = \lambda < 1
\]

1) But \( 0 < \lambda < 1 \)
We have \( \bar{x} = 0 \)
\[
\Rightarrow \left. \frac{dF}{dx} \right|_{x=x_0} = \lambda < 1
\]

2) Consider \( 1 < \lambda < 3 \)
Then \( \bar{x} = 1 - \frac{1}{\lambda} \)
\[
\therefore \left. \frac{dF}{dx} \right|_{x=x_0} = \lambda \left[ 1 - 2 \left( 1 - \frac{1}{\lambda} \right) \right] \\
= \lambda \left[ 1 - 2 + \frac{2}{\lambda} \right] \\
= -\lambda + 2
\]
\[
\Rightarrow \left. \frac{dF}{dx} \right|_{x=x_0} < 1
\]

3) Consider \( 3 < \lambda < 4 \)
We have fixed points
\[
\bar{x} = \frac{1 + \lambda}{2\lambda} \left( 1 \pm \sqrt{\frac{\lambda - 3}{\lambda + 1}} \right)
\]
\[
\Rightarrow \left. \frac{dF}{dx} \right|_{x=x_0} = \lambda \left[ 1 - \frac{1 + \lambda}{\lambda} \left( 1 \pm \sqrt{\frac{\lambda - 3}{\lambda + 1}} \right) \right] \\
= \lambda - \left( 1 + \lambda \right) \left( 1 \pm \sqrt{\frac{\lambda - 3}{\lambda + 1}} \right)
\]
\[
\Rightarrow \left. \frac{dF}{dx} \right|_{x=x_0} < 1
\]

Consider \( x_{n+1} = \lambda x_n (1-x_n) \)
Consider a value of \( x_n \), which is very close to \( \bar{x} \)
Let \( x_n = \bar{x} + \delta_n \) \( \bar{x} \) is a fixed point

Substituting,
\[
\bar{x} + \delta_{n+1} = \lambda \left( \bar{x} + \delta_n \right) (1-\bar{x} - \delta_n) \\
= \lambda \bar{x} + \delta_n - \bar{x} \bar{x} - \delta_n \bar{x} - \delta_n - \delta_n \delta_n
\]

Since \( \delta_n \) is small \( \delta_n^2 \) is negligible.
\[
\bar{x} + \delta_{n+1} = \lambda \bar{x} + \delta_n - \bar{x} \bar{x} - a \delta \bar{x}
\]
\[
\therefore \bar{x} \text{ is fixed point} \]
Then by definition, \( \bar{x} \)
\[
x_{n+1} = x_n = \bar{x}
\]
\[ \therefore \bar{x} = \lambda \bar{x}(1 - \bar{x}) \]
\[ \bar{x}(1 - \lambda (1 - \bar{x})) = 0 \]
\[ \delta_{n+1} = \lambda \left( \bar{x} - \bar{x}^2 + \delta_n - 2\delta_n \bar{x} \right) - \bar{x} \]
\[ \frac{\delta_{n+1}}{\delta_n} = \frac{\lambda \left( 1 - \bar{x} - 2\delta_n \bar{x} \right) - \bar{x}}{\delta_n} \]
\[ = -\bar{x}[-\lambda (1 - \bar{x}) + 1] - 2\lambda \bar{x} \delta_n + \lambda \delta_n \]
\[ = \frac{-\bar{x}[-\lambda (1 - \bar{x}) + 1]}{\delta_n} - 2\lambda \bar{x} \]
\[ \frac{\delta_{n+1}}{\delta_n} = \lambda - \alpha \lambda \bar{x} \]
\[ = \lambda (1 - 2\bar{x}) \]
\[ \therefore \frac{\delta_{n+1}}{\delta_n} = \lambda (1 - 2\bar{x}) = \frac{dF}{dx} \]

If \( \frac{\delta_{n+1}}{\delta_n} < 1 \)
\[ \Rightarrow \delta_{n+1} < \delta_n \]

In other words as \( x_n \) approaches the fixed point \( \bar{x} \) within the difference \( \delta_n, x_{n+1} \) also approaches \( \bar{x} \) with difference less than \( \delta_n \). It is means it is a converging process and after several steps \( x_{n+1} x_n \) becomes same as \( \bar{x} \). Thus \( \bar{x} \) must be the fixed point called as attractor.

If \( \frac{\delta_{n+1}}{\delta_n} > 1 \)
\[ \Rightarrow \delta_{n+1} > \delta_n \]

As \( x_n \) approaches \( \bar{x}, x_{n+1} \) goes away (diverges) further and further from \( \bar{x} \), it is not a fixed point. It is called a repeller.

**Geometric Interpretation:**

![Diagram](image)

**Problem 2)** For the iterated quadratic map \( x_{n+1} = F(F(x_n)) = F_2(x_n) \) Show that its fixed point satisfy
\[ \bar{x}^3 - 2\bar{x}^2 + \left( \frac{\lambda + 1}{\lambda} \right) \bar{x}^2 - \left( \frac{\lambda^2 - 1}{\lambda^3} \right) = 0 \]

Solution:
\[ F_2(x_n) = x_{n+2} = \lambda x_{n+1}(1 - x_{n+1}) \]
\[ x_{n+2} = \lambda [2x_n(1 - x_n)](1 - [2x_n(1 - x_n)]) \]

Let \( x_{n+2} = x_n = \bar{x} \)
\[ x = \alpha \left[ 2 \pi (1 - x) \right] \left( 1 - [2 \pi (1 - x)] \right) \]

\[ x = \alpha \left[ x - x^2 \right] \left( 1 - \alpha \pi + \alpha x \right) \]

\[ x = \alpha \left[ x (1 - x) \right] \left( 1 - \alpha \pi + \alpha x^2 \right) \]

\[ x = \alpha ^2 \left[ 1 - \alpha \pi + \alpha x^2 - x + \alpha x - \alpha x^3 \right] \]

\[ x = \alpha ^2 \left[ 1 - \alpha \pi + \alpha x^2 - x + \alpha x - \alpha x^3 + \alpha x^4 \right] \]

\[ x = \alpha ^2 + x - \alpha x^2 + \alpha x - \alpha x^3 + \alpha x^4 = 0 \]

\[ x = \alpha ^2 + x - \alpha x^2 + \alpha x - \alpha x^3 + \alpha x^4 = 0 \]

\[ x = \alpha ^2 + x - \alpha x^2 + \alpha x - \alpha x^3 + \alpha x^4 = 0 \]

\[ x = \alpha ^2 + x - \alpha x^2 + \alpha x - \alpha x^3 + \alpha x^4 = 0 \]

\[ \Rightarrow x = 0 \]

\[ 1 - \alpha ^2 + \alpha ^2 x - \alpha ^2 x^2 + \alpha ^2 x - \alpha ^2 x^3 = 0 \]

\[ \alpha ^2 x - 2 \alpha x^2 + \left( \alpha ^2 + \alpha x \right) x + 1 - \alpha ^2 = 0 \]

\[ x^3 - 2 \alpha x^2 + \left( \frac{\alpha + 1}{\alpha} \right) x - \left( \frac{\alpha - 1}{\alpha} \right) = 0 \]

In addition to \( x = 0 \), there are three other cubic roots of above equation. Out of these three roots one value is

\[ x = 1 - \frac{1}{\alpha} \]

Factorise this and find the other two values of \( x \)

\[ x = \alpha \left[ x - \alpha \left( x + \alpha \right) \left( x + \beta \right) \left( x + \gamma \right) \right] = 0 \]

\[ \left( x - \gamma \right) \left( x - \beta \right) \left( x - \alpha \right) = 0 \]

\[ \left( x - \gamma \right) \left( x - \beta \right) \left( x - \alpha \right) = 0 \]

\[ x^3 - 2 \alpha x^2 + \left( \frac{\alpha + 1}{\alpha} \right) x + \left( \frac{\alpha - 1}{\alpha} \right) = 0 \]

\[ x^3 - 2 \alpha x^2 + \left( \frac{\alpha + 1}{\alpha} \right) x + \left( \frac{\alpha - 1}{\alpha} \right) = 0 \]

Fracal Dimensions:
The general objects such as point, line, square, cube has integral dimensions. But, there are some geometric constructions for which the dimensions are fractional and not integral.

In general, dimension \( d \) of a set of points in ‘p’ dimensional space is defined as.

\[ d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln \left( \frac{1}{\varepsilon} \right)} \]

When \( N(\varepsilon) \) is the number of \( p \) dimensional cubes of side \( \varepsilon \) needed to cover the entire set. This calculation of dimension is explained as follows

1) Consider a point only point cube is of side \( \varepsilon \) requiring filling the given point.

\[ \therefore N(\varepsilon) = 1 \]

\[ \Rightarrow \ln N(\varepsilon) = 0 \]

\[ \Rightarrow d = 0 \]

2) A line of length ‘l’. This line of length ‘l’ can be filled with small line segments of length \( \varepsilon \)

\[ \therefore N(\varepsilon) = \frac{l}{\varepsilon} \]
\[ d = \lim_{\varepsilon \to 0} \frac{\ln \left( \frac{1}{\varepsilon} \right)}{\ln \left( \frac{1}{\varepsilon} \right)} \]
\[ = \lim_{\varepsilon \to 0} \left[ \ln \left( \frac{1}{\varepsilon} \right) \right] \]
\[ = \lim_{\varepsilon \to 0} \left[ \ln l + \ln \varepsilon^{-1} \right] \]
\[ = \lim_{\varepsilon \to 0} \left[ \ln l + \ln \left( \frac{1}{\varepsilon} \right) \right] \]
\[ = \lim_{\varepsilon \to 0} \left[ 1 + \frac{\ln l}{\ln \left( \frac{1}{\varepsilon} \right)} \right] \]
\[ = \frac{1}{2} \quad \therefore \varepsilon \to 0, \frac{1}{\varepsilon} \to \infty, \ln \left( \frac{1}{\varepsilon} \right) \to \infty \]

3) Consider a square of side \( l \)
[Diagram]
\[ \therefore N(\varepsilon) = \frac{l^2}{\varepsilon^2} \]
\[ d = \lim_{\varepsilon \to 0} \frac{\ln \left( \frac{l^2}{\varepsilon^2} \right)}{\ln \left( \frac{1}{\varepsilon} \right)} \]
\[ = \lim_{\varepsilon \to 0} \left[ \ln l^2 - \ln \varepsilon^2 \right] \]
\[ = \lim_{\varepsilon \to 0} \left[ \ln l + \ln \left( \frac{1}{\varepsilon} \right) \right] \]
\[ = \lim_{\varepsilon \to 0} \left[ \ln l + \ln \left( \frac{1}{\varepsilon} \right) \right] \]
\[ = \lim_{\varepsilon \to 0} \left[ 1 + \frac{\ln l}{\ln \left( \frac{1}{\varepsilon} \right)} \right] \]
\[ = \frac{1}{2} \quad \therefore \varepsilon \to 0, \frac{1}{\varepsilon} \to \infty, \ln \left( \frac{1}{\varepsilon} \right) \to \infty \]

4) Consider a cube of side \( l \)
[Diagram]
So far the dimension of the objects are integral consider the following construction in a set called as "Cantor Set".

Cantor Set is defined as follows:
Consider a line of unit length. Divide it into three equal parts. Then the middle third is removed leaving the two line segments. The process is continued by removing at each step the middle third of each segment.

To cover this set at step greater than 'k'
\[ N(k) = 2^k \]
In each case the side of the cube i.e.
\[ \mathcal{E} = \left( \frac{1}{3} \right)^k \]
\[ d = \lim_{k \to \infty} \frac{\ln 2^k}{\ln 3^k} = \lim_{k \to \infty} \frac{\ln 2}{\ln 3} d = 0.6309 \]

Henon Map:
In many of the examples such as Cantor set, scale invariance was observed. Invariably in scale invariant constructions fractional dimensions was observed. Another scale invariant example is the two dimensional Henon map:
\[ x_{n+1} = 1 - cx_n^2 + y_n \]
\[ y_{n+1} = \beta y_n \]
If the parameter \( \beta = 0 \), then the Henon map reduces to quadratic equation or logistic map. The Henon map consists of two-dimensional curves covering only part of two-dimensional space for \( c = 1.4 \) and \( \beta = 0.3 \) the variation of \( y \) with respect to \( x \) is shown in the following graph.
In figure (a) a small rectangle is shown. In figure (b) the same small rectangle is expanded. Figure (b) also has a small rectangle which is enlarged to figure (c) and so on. All expansions show identical pattern. This is what is called as scale invariance. Wherever scale invariance is observed the construction or set has fractional dimension. The dimension in Henon’s map case is $d = 1.264$.

Lyapunov Exponent

It is known that the non-linear systems are very sensitive to initial conditions. Whether the given system exhibits non-linearity or not can be measured with a coefficient called as Lyapunov exponent. Consider two particles with $d_0$ as initial separation between them. As time changes the dist may also change. The dist of separation between the two particles at any time $t$ is given by:

$$d(t) = d_0 e^{Lt}$$

Where, $L$ is called Lyapunov exponent.

Case (i) If $L \leq 0$

The distance remains same, or keeps on decreasing, it converges to zero after sufficiently long interval of time. We say that the system exhibits linear characteristic. It means the system is not very sensitive to the initial conditions.

ii) If $L > 0$

As time advances the distance of separation keeps on increasing exponentially. The system is said to be non-linear. It is very sensitive to the initial conditions. Thus, the Lyapunov exponent is a measure of whether the system is linear or non-linear.

Approximate analytic steady state solution of Duffing’s oscillator:

The reduced equation of motion of a Duffing’s oscillator is

$$\ddot{x} + 2\gamma \dot{x} + x + x^3 = f \cos \omega t$$

The anharmonic oscillator is subject to periodic force. The system takes some time to respond fully to the applied periodic force. We consider the solutions when the system is stabilized or after sufficiently long interval of time when the initial transients are died down.

The trial solution of a non-linear equation is assumed to be a linear combination of the applied periodic force and its derivative.

Let $x(t) = A(\omega)\cos(\omega t - \theta(\omega))$

$A$ is the amplitude of Duffing’s oscillator and $\theta$ is the phase. These two arbitrary constants depend on $\omega$. 


Substituting this in the above equation we get,
\[ \ddot{x} = -A \cos(\omega t - \theta) \]
\[ \ddot{x} = -A \omega^2 \cos(\omega t - \theta) \]
\[ \therefore -A \omega^2 \cos(\omega t - \theta) - 2\gamma A \omega \sin(\omega t - \theta) + A \cos(\omega t - \theta) \pm A^3 \cos^3(\omega t - \theta) = f \cos \omega t \]
\[ A(1 - \omega^2) \cos(\omega t - \theta) - 2\gamma A \omega \sin(\omega t - \theta) \pm A^3 \cos^3(\omega t - \theta) = f \cos \omega t \]

This equation must be true for all values of time 't'.

We use the following identities:
\[ \cos^3(\omega t - \theta) = \frac{1}{4} \cos(3\omega t - \theta) + \frac{3}{4} \cos(\omega t - \theta) \]
\[ \cos(\omega t - \theta) = \cos((\omega t - \theta) + \theta) \]
\[ = \cos(\omega t - \theta) \cos \theta - \sin(\omega t - \theta) \sin \theta \]

Substituting this in above equation we get,
\[ A(1 - \omega^2) \cos(\omega t - \theta) - 2\gamma A \omega \sin(\omega t - \theta) \pm \frac{A^3}{4} \cos(3\omega t - \theta) \]
\[ = F \cos(\omega t - \theta) \cos \theta - F \sin(\omega t - \theta) \sin \theta \]
\[ \left[ A(1 - \omega^2) - F \cos \theta \pm \frac{3}{4} A^3 \right] \cos(\omega t - \theta) + \left[ F \sin \theta - 2\gamma A \omega \right] \sin(\omega t - \theta) \pm \frac{1}{4} A^3 \cos^3(\omega t - \theta) = 0 \]

As the above relation is true for all values of time the coefficients of \( \cos(\omega t - \theta) \sin(\omega t - \theta) \) and \( \cos 3(\omega t - \theta) \) must be independently equal to zero.

\[ A(1 - \omega^2) - f \cos \theta \pm \frac{3}{4} A^3 = 0 \]
\[ \left[ A(1 - \omega^2) + \frac{3}{4} A^3 \right] - F \cos \theta = 0 \]
\[ -2\gamma A \omega + f \sin \theta = 0 \]
\[ \pm \frac{1}{4} A^3 = 0 \]

Third condition is difficult to obey as it makes \( A = 0 \) and the entire solution does not exist. Therefore we make approximations here we assume \( A \) to be very small so that \( A^3 \) vanishes but \( A \) is still non-zero. The first two equations give us

\[ f \cos \theta = A \left( 1 - \omega^2 \right) \pm \frac{3}{4} A^3 \]
\[ f \sin \theta = 2\gamma A \omega \]

\( \theta \) can be eliminated from the above equation by squaring and adding the above equation.

\[ f^2 = A^2 \left[ \left( 1 - \omega^2 \right) \pm \frac{3}{4} A^2 \right] ^2 + (2\gamma A \omega)^2 \]
\[ \therefore A = \frac{f}{\sqrt{1 - \omega^2 \pm \frac{3}{4} A^2} \pm (2\gamma A \omega)^2} \]

Substitute the value of \( A \) in any of the above equation it can be solved for \( \theta \).

It is obvious that if \( |f(t)| < 1 \) then the \( x^3 \) term in the equation of motion of Duffing’s oscillator vanishes and we get, damped forced oscillator.

For a given frequency \( \omega \), \( f^2 \) is cubic in \( A^2 \) [i.e. \( (A^2)^3 \)] and gives one or three real values for \( A^2 \). Alternatively, by regarding the equation as determining \( \omega^2 \) for given \( A \), a quadric equation results which is easier to solve.
For the following graph we plot \( A \) versus \( \omega \) for the case of hard spring with the damping coefficients \( \gamma = 0.1 \) and force constant \( f = 0.5 \). The dashed curve is the amplitude frequency relation for undamped free oscillator (i.e. \( \gamma = 0, f = 0 \))

We have \( \omega = \sqrt{\frac{3}{4} A^2} \)

Or \( A(\omega) = \left[ \pm \frac{4}{3} (\omega^2 - 1) \right]^{1/2} \)

This curve is called the spine of the resonance. It is the locus of resonance peak as \( f \) is varied in the \( \gamma \to 0 \) limit. The anharmonic cubic term causes the resonance amplitude term to lean over and if \( f \) is sufficiently large then \( A \) and \( \theta \) become triple valued over an interval in \( \omega \).

The middle value in fact corresponds to an unstable steady motion. The phase angle \( \theta \) of \( \omega \) calculated from the previous equations is shown in the following graph.

Dynamics of rigid body

Angular momentum and angular velocity of a rigid body:
Consider, a simple system of two particles attached at the ends of a massless rod i.e. distance between the two particles always remains constant. Let the system be dumbbell shaped rotating about an axis passing through the center of rod.

(Diagram)

Angular momentum of the system is given by
\[ \mathbf{\dot{L}} = \mathbf{\dot{L}}_1 + \mathbf{\dot{L}}_2 \]
\[ = \mathbf{\dot{r}}_1 \times \mathbf{\dot{p}}_1 + \mathbf{\dot{r}}_2 \times \mathbf{\dot{p}}_2 \]
\[ \mathbf{\dot{L}} = m_1 \mathbf{\dot{r}}_1 \times \mathbf{v}_1 + m_2 \mathbf{\dot{r}}_2 \times \mathbf{v}_2 \]

We have,
\[ m_1 = m_2 = m \]
\[ |\mathbf{r}_1| = |\mathbf{r}_2| = \frac{1}{2} \]
\[ \mathbf{\dot{r}}_1 = -\mathbf{\dot{r}}_2 = \mathbf{\dot{r}} \]
\[ \mathbf{\dot{v}}_1 = -\mathbf{\dot{v}}_2 = \mathbf{\dot{v}} \]
\[ . \; \mathbf{\dot{L}} = m\mathbf{\dot{r}} \times \mathbf{\dot{v}} [1 + \{-1\}(1)] \]
\[ \mathbf{\dot{L}} = 2m\mathbf{\dot{r}} \times \mathbf{\dot{v}} \]

Obvious from the above relation, that the direction of angular momentum \( \mathbf{\dot{L}} \) is perpendicular to the rod. Therefore, it is not parallel to the direction of angular velocity or axis of rotation. Thus, in general the direction of angular momentum is not along the axis of rotation of the rigid body. Only when \( \theta = 90^\circ \), \( \mathbf{\dot{L}} \) will be parallel to \( \mathbf{\dot{\omega}} \).

Consider an inertial frame. According to Newton’s law of motion we have,
\[ \mathbf{\dot{N}} = \frac{d\mathbf{\dot{L}}}{dt} \]

We introduce a rotating frame such that the rotating rigid body is at rest in this non-inertial frame. In other words, the starred frame rotates with angular velocity \( \mathbf{\omega} \). We have
\[ \left( \frac{d\mathbf{\dot{L}}}{dt} \right)_{\text{rot}} = \left( \frac{d\mathbf{\dot{L}}}{dt} \right)_{\text{rest}} + \mathbf{\omega} \times \mathbf{\dot{L}} \]

But in the rotating frame, rigid body is at rest. Hence, \( \left( d\mathbf{\dot{L}}/dt \right) = 0 \).

Therefore, we get the relation:
\[ \mathbf{\dot{N}} = \mathbf{\dot{\omega}} \times \mathbf{\dot{L}} \]

If \( \mathbf{\dot{\omega}} \) and \( \mathbf{\dot{L}} \) are parallel than \( \mathbf{\dot{\omega}} \times \mathbf{\dot{L}} = 0 \)
This implies \( \mathbf{\dot{N}} = 0 \). Thus, whenever direction of angular momentum is parallel to the axis of rotation, then the body rotates freely i.e. without any torque acting on it.

**Moment of Inertia**

Inertial mass is a measure of opposition to the change in linear motion of the particle. Analogously, moment of inertia is a measure of opposition to the change in angular motion of the rigid body. It not only depends on the mass of the rigid body but also on the distribution of mass about the axis of rotation. Consider a rigid body, (Rigid body consist of large number of particles with the constraints that distance between any two particles is always constant) rotating about an axis passing through a point in the rigid body. Therefore, there is no translational motion of the body but only the rotational motion. Each particle has the same angular velocity \( \mathbf{\dot{\omega}} \) but different linear velocities and the position vectors of all the particles are different.

For \( i \)th particle
\[ \mathbf{\dot{v}}_i = \mathbf{\dot{\omega}} \times \mathbf{\dot{r}}_i \]

The linear momentum of the \( i \)th particle
\[ \mathbf{\dot{p}}_i = m_i \mathbf{\dot{v}}_i \]
Angular momentum of the $i$th particle

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i$$

$$\vec{L}_i = m_i \vec{r}_i \times \vec{v}_i$$

Hence Angular momentum of the rigid body is

$$\vec{L} = \sum_i \vec{L}_i$$

$$= \sum_i m_i (\vec{r}_i \times \vec{v}_i)$$

$$= \sum_i m_i (\vec{r}_i \times (\vec{\omega} \times \vec{r}_i))$$

$$= \sum_i m_i [\vec{\omega}(\vec{r}_i \cdot \vec{\omega}) - \vec{r}_i (\vec{\omega} \cdot \vec{\omega})]$$

$$= \sum_i m_i r_i^2 \vec{\omega} - \sum_i m_i (\vec{r}_i \cdot \vec{\omega}) \vec{r}_i$$

In a Cartesian co-ordinate system we have,

$$\vec{r}_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$$

$$\Rightarrow r_i^2 = x_i^2 + y_i^2 + z_i^2$$

and $$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\therefore \vec{L} = \sum_i m_i \left( x_i^2 \omega_y + y_i^2 \omega_x + z_i^2 \omega_x \right) - \sum_i m_i \left( x_i \omega_z + y_i \omega_x + z_i \omega_x \right) \left( x_i \hat{i} + y_i \hat{j} + z_i \hat{k} \right)$$

Let $$\vec{L} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$$

$$\Rightarrow L_x = \sum_i m_i \left( y_i^2 + z_i^2 \right)$$

$$L_y = \sum_i m_i \left( x_i^2 + z_i^2 \right)$$

$$L_z = \sum_i m_i \left( x_i^2 + y_i^2 \right)$$

$$I_{xx} = -\sum_i m_i x_i y_i = I_{yy}$$

$$I_{yy} = -\sum_i m_i x_i z_i = I_{zz}$$

$$I_{zz} = -\sum_i m_i y_i z_i = I_{xx}$$

$$\therefore L_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z$$

$$L_y = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z$$

$$L_z = I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z$$

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$L = \text{Moment of inertia tensor}$$

The component $L_{xy}, L_{yz}, L_{zx}$ is called products of moment of inertia of rigid body.
The components $I_{xx}$, $I_{yy}$, $I_{zz}$ are called components of moments of inertia of rigid body.

The angular momentum of rigid body can now be expressed as
$$\vec{L} = \vec{I} \cdot \vec{\omega}$$

Where $\vec{I}$ is the moment of inertia tensor having 9 components in general.

**Principle axis and principle momentum:**

For a regular symmetric bodies it is always possible to select the special coordinate system i.e. principal axis such that products of moments of inertia vanish, only the components $I_{xx}$, $I_{yy}$ and $I_{zz}$ are non-zero. These non-zero components are called principle moments of inertia. Therefore, for principle axis the angular momentum of a rigid body can be expressed as
$$\vec{L} = I_{xx}\hat{i} \hat{\omega} + I_{yy}\hat{j} \hat{\omega} + I_{zz}\hat{k} \hat{\omega}$$

**Euler’s equation**

Consider a rigid body rotating about an axis passing through some point in the rigid body. Let ‘$\vec{L}$’ be the angular momentum of the rigid body. According to Newton’s 2nd law of motion, the torque acting on the rigid body should be same as rate of change of angular momentum of the rigid body. In general, the angular momentum of the rigid body is given by,
$$\vec{N} = \frac{d\vec{L}}{dt}$$

This is the set of Euler’s equation for rigid body motion.

**Classification of rigid bodies:**

Depending on the symmetry involved in the distribution of mass in a rigid body, there are different types of rigid bodies:

1. $I_x \neq I_y \neq I_z$ (asymmetric top)
2. $I_x = I_y \neq I_z$ (spherical top)
3. $I_x = I_y = I_z$ (symmetric top)
4. $I_x = I_y, I_z = 0$ (Rotor)

**Symmetric Top**

It is difficult to solve Euler’s equation for the symmetric top under the application of non-zero torque, as it is difficult to express torque in terms of body axis. But, it is worth discussing ‘torque free’ motion of the symmetric top.

Since there is no torque acting on the symmetric top i.e. $\vec{N} = 0$ we have

$$N_x = I_x \omega_x + (I_z - I_y) \omega_y \omega_z = 0$$
$$N_y = I_y \omega_y + (I_z - I_x) \omega_x \omega_z = 0$$
$$N_z = I_z \omega_z + (I_y - I_x) \omega_x \omega_y = 0$$

Where $I_x$, $I_y$, and $I_z$ are the principle moments of inertia of the symmetric top about the body principle axis.

For symmetric top,

$L_x = L_y 

Therefore, above three equations simplify as

Let $L_x = L_y$

$$I_x \omega_x + (I_z - I_y) \omega_y \omega_z = 0 \quad \text{A}$$
$$I_y \omega_y + (I_z - I_x) \omega_x \omega_z = 0 \quad \text{B}$$
$$I_z \omega_z + (I_y - I_x) \omega_x \omega_y = 0 \quad \text{C}$$

$$\Rightarrow I_z \omega_z = 0$$
The third equation implies that $\dot{\omega}_z = 0 \Rightarrow \omega_z = \text{const}$

Simplifying $A$ and $B$

$$\dot{\omega}_z + \left[ \frac{I_z - I_x}{I_z} \right] \omega_z = 0$$

$$& \dot{\omega}_y = \left[ \frac{I_x - I_z}{I_x} \right] \omega_y = 0$$

Let $\left( \frac{I_x - I_z}{I_z} \right) \omega_z = \Omega$

$$\therefore \dot{\omega}_z + \Omega \omega_x = 0$$

$$\dot{\omega}_y - \Omega \omega_y = 0$$

The above two differential equations are interdependent coupled ordinary differential equations in $\omega_z$ and $\omega_y$. They can be solved easily as follows:

Differentiating the first equation with respect to time we get,

$$\dot{\omega}_z + \Omega \omega_x = 0$$

Substituting from 2nd equation we get,

$$\dot{\omega}_z + \Omega^2 \omega_x = 0$$

Similarly we get,

$$\dot{\omega}_y + \Omega^2 \omega_y = 0$$

This is 2nd order, linear, ordinary differential equation with constant coefficient.

The solutions are:

$$\omega_z = A \cos(\Omega t + \theta_0)$$

$$\omega_y = -A \Omega \sin(\Omega t + \theta_0)$$

Substituting in 1st equation we get,

$$\omega_z = A \sin(\Omega t + \theta_0) + \Omega \omega_y = 0$$

$$\omega_z^2 + \omega_y^2 = A^2 \text{ (constant)}$$

$$\therefore \omega_z^2 = A^2$$

$$\Rightarrow |\omega| = \text{constant}$$

Thus, the magnitude of an angular velocity of the symmetric top is constant. This angular velocity vector keeps on rotating about the $z$ axis so that its $z$-component remains constant as shown in the diagram.

(Diagram)

$$\dot{\Omega} = \left( \frac{I_z - I_x}{I_x} \right) \dot{\omega}_z$$

If $I_z > I_x$, then direction of $\dot{\Omega}$ is same as that of $\dot{\omega}_z$.

If $I_z < I_x$, then the direction of $\Omega$ is opposite to that of $\dot{\omega}_z$.

It implies that the symmetric top makes slow precessional motion with angular velocity $\Omega$ about the body $Z$-axis making an angle $\alpha_z$ with the $Z$-axis. The body axis sweeps a cone about the $Z$-axis. This cone formed is called a body cone.

The solutions thus obtained refer to the body axis i.e. rotating frame. But, for the observer fixed with respect to the space can be understood as the symmetric top rotates about its own axis and subtends angle $\alpha_s$ is space with the body axis. The symmetric top also sweeps a cone in space, it is called space
cone. The body cone and a space cone of the symmetric top are shown in the following diagram:

(Diagram)

Euler’s angle \((\phi, \theta, \psi)\)

Rigid body has six degrees of freedom. Three co-ordinates are required to specify the point in the rigid body say centre of mass. Remaining three co-ordinates are required to specify the orientation of the rigid body. Three angles \((\phi, \theta, \psi)\) are used to specify the orientations of the rigid body. They are called Euler’s angles.

Starting from any orientation of a rigid body, any new general orientation can be obtained through three independent rotations of the rigid body.

\[
(x, y, z) \underbrace{\text{about } Z (x', y', z')}_{\phi} \underbrace{\text{about } X'}_{\theta} \underbrace{\text{about } Z'}_{\psi} (1, 2, 3)
\]

(Diagram)

Corresponding to the three Euler’s angles there will be three types of angular velocities \(\phi, \theta, \psi\). The direction of \(\phi\) is along the old \(Z\)-axis whereas the direction of \(\theta\) is along the new \(X'\) axis whereas \(\psi\) is along the new \(Z\)-axis as shown in the following diagram. The angular velocities are always along the axis of rotation.

(Diagram)

The angular velocity \(\omega\) of the rigid body can now be expressed in terms of the components of angular velocity along the body principal axes \((1, 2, 3)\). Thus, we have

\[
\omega = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3
\]

Where,

\[
\omega_1 = \dot{\theta} \cos \psi + \phi \sin \theta \sin \psi
\]

\[
\omega_2 = \phi \sin \theta \cos \psi - \dot{\theta} \sin \psi
\]

\[
\omega_3 = \dot{\psi} + \phi \cos \theta
\]

Symmetric Top with torque acting on it

(Diagram)

The torque acting on the symmetric top \(N = mgl \sin \theta\)

For symmetric top, \(I_1 = I_2 \neq I_3\)

The Kinetic Energy of the symmetric top is given by,

\[
K = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2
\]

But \(I_1 = I_2\)

\[
= \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2
\]

\[
= \frac{1}{2} I_1 \left[ (\dot{\theta} \cos \psi + \phi \sin \theta \sin \psi)^2 + (\phi \sin \theta \cos \psi - \dot{\theta} \sin \psi)^2 + \frac{1}{2} \dot{\psi} (\dot{\psi} + \phi \cos \theta)^2 \right]
\]
\[ \frac{1}{2} I_1 \left[ \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right] + \frac{1}{2} I_2 \left[ \ddot{\psi} + \dot{\phi} \cos \theta \right]^2 \]

Potential Energy \( V = mgl \cos \theta \)

\[ :: L = K - V \]

\[ L = \frac{1}{2} I_1 \left[ \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right] + \frac{1}{2} I_2 \left[ \ddot{\psi} + \dot{\phi} \cos \theta \right]^2 - mgl \cos \theta \]

\[ :: \text{Lagrange equation} \]

1) \[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \]

\[ \frac{\partial L}{\partial \dot{\theta}} = I_1 \dot{\theta} \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = I_1 \ddot{\theta} \]

\[ \frac{\partial L}{\partial \theta} = I_1 \dot{\phi}^2 \sin \theta \cos \theta - I_2 \left( \dot{\psi} + \dot{\phi} \cos \theta \right) \phi \sin \theta + mgl \sin \theta \]

\[ \therefore I_1 \ddot{\theta} - I_1 \dot{\phi}^2 \sin \theta \cos \theta - I_2 \left( \dot{\psi} + \dot{\phi} \cos \theta \right) \phi \sin \theta + mgl \sin \theta = 0 \]

2) \[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0 \]

\[ \rho_\phi = \frac{\partial L}{\partial \phi} = \text{constant} \]

\[ \rho_\phi = I_1 \dot{\phi} \sin \theta \cos \theta + I_2 \left( \dot{\psi} + \dot{\phi} \cos \theta \right) \cos \theta \]

3) \[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\psi}} \right) - \frac{\partial L}{\partial \psi} = 0 \]

\[ \rho_\psi = \frac{\partial L}{\partial \psi} = \text{constant} \]

\[ \rho_\psi = I_1 \left( \dot{\psi} + \dot{\phi} \cos \theta \right) \]

Central Force

Central force is force acting on the particle such that it depends only on the distance of the particle from a fixed point and the direction of a force is always along the line joining the particle and the fixed point. The fixed point is called the center of force.

\( \vec{F} = f(r) \hat{r} \)

e.g.

1) Gravitational force of attraction

\[ \vec{F} = -G \frac{m_1 m_2}{r^2} \hat{r} \]

\[ = \frac{k}{r^2} \hat{r} \]

(Diagram)

2) Electrostatic force between two charge particles

\[ \vec{F} = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r} \]
Consider a particle acted upon by a central force. The angular momentum of a particle is given by
\[ \mathbf{L} = \mathbf{r} \times \mathbf{p} \]

Take center of force as origin of co-ordinate system. \( \mathbf{p} \) is the linear momentum of particle.

\[ \mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m \mathbf{v} = m \mathbf{r} \times \frac{d\mathbf{r}}{dt} \]

Differentiating with respect to time
\[ \frac{d\mathbf{L}}{dt} = m \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} + m \mathbf{r} \times \frac{d^2 \mathbf{r}}{dt^2} \]
\[ = 0 + \mathbf{r} \times m \frac{d^2 \mathbf{r}}{dt^2} \]
\[ = \mathbf{r} \times \mathbf{F}(r) \frac{d\mathbf{r}}{dt} \]
\[ = \mathbf{r} \times \mathbf{F}(r) \hat{r} \]
\[ = \mathbf{r} \mathbf{F}(r) \left( \frac{d\mathbf{r}}{dt} \times \mathbf{r} \right) \]

\[ \frac{d\mathbf{L}}{dt} = 0 \]
\[ \mathbf{L} = \text{constant} \]

Thus, the angular momentum \( \mathbf{L} \) of the particle is constant; constant in magnitude and constant in direction.

Consider
\[ \mathbf{L} = m \mathbf{r} \times \mathbf{v} \]

Taking dot product with \( \hat{r} \)
\[ \hat{r} \cdot \mathbf{L} = m \mathbf{r} \cdot \hat{r} \times \mathbf{v} \]
\[ = m [\mathbf{r} \cdot (\hat{r} \times \mathbf{v})] \]
\[ = 0 \quad \therefore \hat{r} \cdot (\hat{r} \times \mathbf{v}) = 0 \]

This implies that the angular momentum of a particle experiencing a central force must be perpendicular to the position vector of the particle.

Consider
\[ \mathbf{v} \cdot \mathbf{L} = m [\mathbf{v} \cdot (\hat{r} \times \mathbf{v})] \]
\[ = 0 \]

This implies that the angular momentum of a particle is always perpendicular to the velocity of the particle. In other words the particle moves in such a way that its velocity always remain perpendicular to direction of angular momentum.

It is obvious that the angular momentum \( \mathbf{L} \) is always perpendicular to the plane containing the radius vector and the velocity of the particle. Since angular momentum is constant in direction and magnitude the plane containing \( \hat{r} \times \mathbf{v} \) must also be fixed.

Thus, a particle moving under the influence of a central force always perform a motion confined to a single plane.
Plane polar co-ordinate

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

Since position vector of \( \vec{r} \) is,
\[ \vec{r} = x \hat{i} + y \hat{j} \]
\[ \vec{r} = r \cos \theta \hat{i} + r \sin \theta \hat{j} \]

We have,
\[ \dot{\vec{r}} = \frac{\partial \vec{r}}{\partial r} \frac{dr}{dt} + \frac{\partial \vec{r}}{\partial \theta} \frac{d\theta}{dt} \]
\[ \dot{\vec{r}} = \cos \theta \hat{i} + \sin \theta \hat{j} \]
\[ \dot{\theta} = \frac{1}{r} \frac{\partial \vec{r}}{\partial \theta} \]
\[ = \frac{1}{r} \left( - \sin \theta \hat{i} + \cos \theta \hat{j} \right) \]
\[ \dot{\theta} = - \sin \theta \hat{i} + \cos \theta \hat{j} \]

\[ \vec{r} = r \dot{\vec{r}} \]
\[ \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt} \left( r \dot{\vec{r}} \right) \]
\[ = \frac{dr}{dt} \dot{r} + r \left( \frac{d\dot{r}}{dt} \right) \]
\[ = \frac{dr}{dt} \dot{r} + r \left( \frac{d\dot{r}}{d\theta} \frac{d\theta}{dt} \right) \]
\[ = \dot{r} \dot{\theta} \sin \theta \hat{i} + \cos \theta \hat{j} \]
\[ = \dot{r} \dot{\theta} \]
\[ \vec{v} = v_r \hat{r} + v_\theta \hat{\theta} \]
\[ \Rightarrow \ v_r = \dot{r} \quad \text{and} \quad v_\theta = r \dot{\theta} \]

\[ \ddot{a} = \frac{d\vec{v}}{dt} \]
\[ = \frac{d}{dt} \left( r \dot{\vec{r}} + r \dot{\theta} \hat{\theta} \right) \]
\[ = \ddot{r} \dot{\theta} \hat{r} + r \dot{r} \dot{\theta} \hat{\theta} + r \dot{\theta} \dot{\theta} \hat{r} + r \dot{\theta} \dot{\theta} \hat{\theta} \]
\[ = \ddot{r} \dot{\theta} \hat{r} + \dot{r} \dot{\theta} \hat{\theta} + \dot{\theta} \dot{\theta} \hat{r} + \dot{\theta} \dot{\theta} \hat{\theta} \]
\[ = \ddot{r} \dot{\theta} \hat{r} + \dot{r} \dot{\theta} \hat{\theta} + \dot{\theta} \dot{\theta} \hat{r} + \dot{\theta} \dot{\theta} \hat{\theta} \]
\[ = (r - r \dot{\theta}^2) \ddot{r} + (2r \dot{\theta} + r \dot{\theta}) \ddot{\theta} \]
Let \( a = a_r \hat{r} + a_\theta \hat{\theta} \)
\[
\begin{align*}
a_r &= \ddot{r} - r \dot{\theta}^2 \\
a_\theta &= 2 r \dot{\theta} + r \ddot{\theta}
\end{align*}
\]
\[
F = m \ddot{a} = \big( m \ddot{r} - m r \dot{\theta}^2 \big) \hat{r} + \big( m r \ddot{\theta} + 2 m r \dot{\theta} \big) \hat{\theta}
\]
\[
F = F_r \hat{r} + F_\theta \hat{\theta}
\]
\[
\Rightarrow F_r = m \ddot{r} - m r \dot{\theta}^2
\]
\[
F_\theta = m r \ddot{\theta} + 2 m r \dot{\theta}
\]

Using above mathematics, it can be shown that a particle undergoing motion under the influence of central force, its total angular momentum is always conserved.

For a particle of mass ‘\( m \)’ the angular momentum is given by
\[
L = l \omega = m r^2 \dot{\theta}
\]

Consider
\[
\frac{dL}{dt} = \frac{d}{dt} \big( m r^2 \dot{\theta} \big)
\]
\[
= m \frac{d}{dt} \big( r^2 \dot{\theta} \big)
\]
\[
= m 2 r r \ddot{\theta} + m r^2 \dot{\theta}
\]
\[
= m r \big[ 2 \ddot{r} \dot{\theta} + r \ddot{\theta} \big]
\]
\[
= r \big[ m ( \ddot{r} + r \ddot{\theta} ) \big]
\]
\[
= r \big[ m a_\theta \big]
\]
\[
r F_\theta
\]

Since particle experiences central force
\[
\ddot{F} = F(r) \hat{r}
\]
\[
\Rightarrow F_\theta = 0
\]

Hence,
\[
\frac{dL}{dt} = 0
\]

Thus, particle moving under the influence of central force, its total angular momentum is always conserved.

The direction of angular momentum can be obtained as follows:
\[
\hat{L} = \hat{r} \times \hat{p}
\]
\[
= \hat{r} \times m \dot{\vec{v}}
\]
\[
= m r \hat{r} \times \big[ \ddot{r} \hat{r} + r \dot{\theta} \hat{\theta} \big]
\]
\[
= m r ( \ddot{\vec{r}} + r \ddot{\theta} \hat{r} \times \hat{\theta} )
\]
\[
= m r ( \ddot{\vec{r}} ) + m r^2 \ddot{\theta} \hat{r} \times \hat{\theta}
\]
\[
= m r ( \ddot{\vec{r}} ) + m r^2 \ddot{\theta} \hat{k}
\]
\[
\hat{\lambda} = m r^2 \hat{k}
\]
\[
\hat{\lambda}
\]

Kepler’s Laws of planetary motion:

Three laws based on observation of the planetary motion are given below and called as Kepler’s laws of planetary motion.

1) The orbit of a planet moving under the influence of central force is a close orbit mostly elliptical. The motion of the planet is periodic.

2) As a planet moves around the Sun, it sweeps equal areas in equal interval of time in other words, if the planet is closed to Sun it moves fast and if it is away from the center, it moves slowly.
3) The cube of a radius of a orbit of a planet is directly proportional to the square of the periodic time i.e. \( R^3 \propto T^2 \)

Proof of the 2nd law: i.e. each planet sweeps an equal area in equal intervals of time.

[Diagram]

The area swept by the radius vector of the particle as it moves from \( P \) to \( Q \) in time \( \Delta t \) is
\[
\Delta S = \frac{1}{2} r |\Delta \vec{r}| = \frac{1}{2} r^2 \Delta \theta
\]
\[
\lim_{\Delta \theta \to 0} \frac{\Delta S}{\Delta \theta} = \frac{dr}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}
\]
\[
= \frac{m r^2 \dot{\theta}}{2m} \quad \text{(Dividing and multiplying by } m) \]
\[
= \frac{L}{2m} \quad \text{Constant}
\]
As angular momentum \( \dot{L} \) is constant, \( dr/dt \) is always constant. It means radius vector sweeps equal area in equal intervals of time.

**Equation of an orbit:**
Consider a particle like planet moving under the influence of central force. The trajectory traced by the particle as a function of time is called the equation of the orbit. The equation of orbit doesn't involve time explicitly.

The force is given by,
\[
\vec{F} = F(r) \dot{\vec{r}}
\]
\[
\Rightarrow F_r = F(r) = m (\ddot{r} - r \dot{\theta}^2)
\]
\[
= m \frac{d^2 r}{dt^2} - m r \left( \frac{d\theta}{dt} \right)^2
\]
But \( L = mr^2 \dot{\theta} \)
\[
L^2 = m^2 r^2 \dot{\theta}^2
\]
\[
\Rightarrow \dot{\theta}^2 = \frac{L^2}{m^2 r^2}
\]
\[
\therefore \frac{d^2 r}{dt^2} - m r \frac{L^2}{m^2 r^3} = F(r)
\]
\[
m \frac{d^2 r}{dt^2} - \frac{L^2}{mr^3} = F(r)
\]
Consider a special case when \( L = 0 \)
\[
\Rightarrow \dot{\theta} = 0 \Rightarrow \dot{\theta} = \text{constant}
\]
Thus, the particle moves along the straight line e.g. \( \alpha \) particles heading towards a heavy gold nucleus.

Consider \( L \neq 0 \)
Let \( r = \frac{1}{u} \) or \( u = \frac{1}{r} \) (Change of variable)
\[
L = mr^2 \dot{\theta}
\]
\[
= m \frac{1}{u^2} \frac{d\theta}{dt}
\]
\[ \frac{d\theta}{dt} = \frac{Lu^2}{m} \]

Consider \[ \frac{dr}{du} = \frac{d}{du}\left(\frac{1}{u}\right) \]
\[ = -\frac{1}{u^2} \]

Consider \[ \frac{dr}{dt} = \frac{dr}{du} \cdot \frac{du}{dt} \]
\[ = \frac{dr}{du} \cdot \frac{d\theta}{dt} \]
\[ = \left( -\frac{1}{u^2} \right) \left( \frac{d\theta}{dt} \right) \left( \frac{Lu^2}{m} \right) \]
\[ \frac{dr}{dt} = -\frac{L}{m} \frac{du}{d\theta} \]

Consider \[ \frac{d^2r}{dt^2} = \frac{d}{dt}\left( \frac{dr}{dt} \right) \]
\[ = \frac{d}{dt}\left( -\frac{L}{m} \frac{du}{d\theta} \right) \]
\[ = \left( -\frac{L}{m} \right) \left( \frac{d}{d\theta} \left( \frac{du}{d\theta} \right) \right) \]
\[ = \left( -\frac{L}{m} \right) \frac{d^2u}{d\theta^2} \left( \frac{Lu^2}{m} \right) \]
\[ = \left( -\frac{L^2u^2}{m^2} \right) \frac{d^2u}{d\theta^2} \]

From equation
\[ \left( -\frac{L^2u^2}{m} \right) \frac{d^2u}{d\theta^2} - \frac{L^2u^2}{m} = F\left( \frac{1}{u} \right) \]
\[ - \frac{d^2u}{d\theta^2} + u = \left( -\frac{m}{L^2u^2} \right) F\left( \frac{1}{u} \right) \]

Consider a special case of central force, where in force is inversely proportional to the square of the distance of the particle from the center i.e. inverse square law.

\[ \therefore F(r) = \frac{k}{r^2} \]
\[ = ku^2 \]
\[ \frac{d^2u}{d\theta^2} + u = -\frac{mk}{L^2} \]

The solution above equation is the complementary solution and the particular integral. In this case, the particular integral itself is constant right hand side. Therefore the solution is
\[ \frac{1}{r} = u = -\frac{mk}{L^2} + A\cos(\theta - \theta_0) \]