B. N. BANDODKAR COLLEGE OF SCIENCE, THANE
S. Y. B. Sc. SEMESTER IV
MATHEMATICS PAPER II (USMT 402)
THEORY QUESTION BANK

UNIT 1: LINEAR TRANSFORMATION

1. Let V and W be vector spaces over $\mathbb{R}$. Then show that $T: V \longrightarrow W$ is a linear transformation if and only if $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$ for all $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{R}$.
2. If $T: V \longrightarrow W$ is a linear transformation, then prove that
   (i) $T(0) = 0$ (ii) $T(-v) = -T(v)$ (iii) $T(v - w) = T(v) - T(w)$ for all $v, w \in V$.
3. Let $T: V \longrightarrow W$ be a linear transformation. Then show that
   (i) $\ker T$ is a subspace of V.
   (ii) $Img\ T$ is a subspace of W.
4. Let $T: V \longrightarrow W$ be a linear transformation. Prove that T is injective if and only if $\ker T = \{0\}$.
5. Let V and W be vector spaces over $\mathbb{R}$ and $\{v_1, v_2, \cdots, v_n\}$ be a basis of V. Let $w_1, w_2, \cdots, w_n$ be any arbitrary elements of W. Then show that there exits a unique linear transformation $T: V \longrightarrow W$ such that $T(v_i) = w_i; i = 1,2, \cdots, n$.
6. Let $T: V \longrightarrow W$ be a linear transformation with $\ker T = \{0\}$. If $v_1, v_2, \cdots, v_n$ are linearly independent elements of V, then prove that $T(v_1), T(v_2), \cdots, T(v_n)$ are linearly independent elements of W.
7. State and prove Rank-Nullity theorem.
8. State Rank-Nullity theorem. Hence prove that, if V and W are two vector spaces such that $\dim V = \dim W$ and if $T: V \longrightarrow W$ is a linear transformation, then show that T is injective iff T is surjective.
9. Let U, V and W be vector spaces over $\mathbb{R}$. Let $T: U \longrightarrow V$ and $S: V \longrightarrow W$ be linear transformations. Prove that the composite map $S\ o\ T$ is also a linear transformation.
10. Let V be an n dimensional vector space with ordered basis $B_1 = \{v_1, v_2, \cdots, v_n\}$ and let W be m dimensional vector space with ordered basis $B_2 = \{w_1, w_2, \cdots, w_m\}$. Then show that there is a one-to-one correspondence between $L(V, W)$ and $M_{m \times n}$.
11. Let V be an n dimensional vector space with ordered basis $B_1 = \{v_1, v_2, \cdots, v_n\}$ and let W be m dimensional vector space with ordered basis $= \{w_1, w_2, \cdots, w_m\}$. Let $T: V \longrightarrow W$ be a linear transformation. Show that $[T(v)]_{B_2} = [m(T)]_{B_1}^{B_2} [v]_{B_1}$ where $[m(T)]_{B_1}^{B_2}$ denotes matrix of T w.r.t bases B_1 and B_2.
12. Let U, V and W be vector spaces over $\mathbb{R}$. Let $T: U \longrightarrow V$ and $S: V \longrightarrow W$ be linear transformations. Then prove that
   (i) $SoT: U \longrightarrow W$ is a linear transformation.
   (ii) $\alpha T: U \longrightarrow V$ is a linear transformation, for $\alpha \in \mathbb{R}$.
13. Let $T: V \longrightarrow W$ be a linear transformation. If T is invertible then prove that T^{-1} is also a linear transformation.

14. Let V and W be vector spaces over $\mathbb{R}$. Let $T, S: V \longrightarrow W$ be linear transformations. Let $B_1$and B_2 be ordered basis of V and W respectively. Then show that
$$[m(S+T)]_{B_1}^{B_2} = [m(S)]_{B_1}^{B_2} + [m(T)]_{B_1}^{B_2} \text{ and}$$
$$[m(cT)]_{B_1}^{B_2} = c[m(T)]_{B_1}^{B_2} \text{ for any } c \in \mathbb{R}.$$
15. Let V_1, V_2 and V_3 be vector spaces over $\mathbb{R}$ with bases B_1, B_2 and B_3 respectively. Let $T: V_1 \longrightarrow V_2$ and S: $V_2 \longrightarrow V_3$ be linear transformations. Then show that
$$[m(S\ o\ T)]_{B_1}^{B_3} = [m(S)]_{B_2}^{B_3} \times [m(T)]_{B_1}^{B_2}.$$
16. Let V be a vector space and $T: V \longrightarrow V$ be a linear transformation. Let $A = [m(T)]_B^B$ where $B = \{v_1, v_2, \cdots, v_n\}$ is an ordered basis of V. Then show that T is invertible iff A is invertible and hence show that $[m(T^{-1})]_B^B = A^{-1}$.
17. Let V be a vector space and let B_1, B_2 be two bases of V. Let $T: V \longrightarrow W$ be a linear transformation. Then prove that there exits an invertible matrix N such that
$$[m(T)]_{B_2}^{B_2} = N^{-1}[m(T)]_{B_1}^{B_1}N.$$

UNIT 2: DETERMINANTS

1. Let $f: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a multilinear, alternating function with $f(I) = 1$. Prove that $f(A^1, A^2) = \det(A^1, A^2)$ where $A^1, A^2 \in \mathbb{R}^2$ are column vectors and I is the identity matrix.
2. Let A be an $n \times n$ matrix. Let i, j be integers with $1 \leq i, j \leq n$ and $i \neq j$. If the i^{th} and j^{th} columns are interchanged, then show that the determinant changes its sign.
3. If a scalar multiple on one column is added to another, show that the determinant does not change.
4. Define determinant using permutations and with usual notations, prove that $\det(A^1, A^2, \cdots, A^j = B + C, \cdots, A^n) = \det(A^1, A^2, \cdots, B, \cdots, A^n) + \det(A^1, A^2, \cdots, C, \cdots, A^n)$
5. Define determinant using permutations and with usual notations, show that $\det(A^1, A^2, \cdots, A^j = \alpha B, \cdots, A^n) = \alpha \det(A^1, A^2, \cdots, B, \cdots, A^n)$.
6. Prove that $\det A = \det A^T$, where A is a $n \times n$ matrix and A^T denotes transpose of A.
7. Let A, B be $n \times n$ matrices. Show that $\det(AB) = \det A \det B$.
8. Let A be a $n \times n$ matrix. Prove that $A \times adj\ A = \det A \times I$, where $adj\ A$ denotes adjoint of matrix A and I denotes $n \times n$ identity matrix.
9. If $A^1, A^2, \cdots, A^n$ are n linearly dependent column vectors in $\mathbb{R}^n$, then prove that $\det(A^1, A^2, \cdots, A^n) = 0$.
10. State and prove Cramer's rule.
11. Let A and B be two similar matrices. Prove that $\det(A) = \det(B)$.
12. Let A be a $n \times n$ matrix. Prove that $\det(adj\ A) = (\det A)^{n-1}$

UNIT 3: EIGENVALUES AND EIGENVECTORS.

1. Define Eigenvalue and characteristic polynomial of a matrix and prove that eigenvalues of a matrix are the roots of the characteristic polynomial.

2. Define similar matrices. Let A and B be two similar matrices. Prove that $\det(A) = \det(B)$.
3. Define characteristic polynomial of a matrix. Show that A and A^T have same characteristic polynomial.
4. Given A and B are similar matrices. Let λ be an eigenvalue of A. Then show that λ is an eigenvalue of B also.
5. If λ is an eigenvalue of A, then prove that λ^k is an eigenvalue of A^k, for all $k \in \mathbb{N}$.
6. Prove that A is invertible iff $\lambda = 0$ is an eigenvalue of A.
7. If λ is an eigenvalue of A, then prove that $\alpha\lambda$ is an eigenvalue of αA, where $\alpha \in \mathbb{R}$.
8. Let A be an $n \times n$ invertible matrix. If λ is an eigenvalue of A, then prove that $\lambda \neq 0$ and λ^{-1} is an eigenvalue of A^{-1}.
9. Let A be a $n \times n$ matrix. Define eigenspace of A and prove that eigenspace is a subspace of $\mathbb{R}^n$.
10. Let λ be an eigenvalue of a $n \times n$ matrix A. Let X and Y be eigenvectors corresponding to eigenvalue λ. Then for any scalars $\alpha, \beta > 0$, show that $\alpha X + \beta Y$ is again an eigenvector corresponding to the eigenvalue λ.
11. If λ is an eigenvalue of A, then for $\alpha, c \in \mathbb{R}$ prove that
    (i) $\alpha\lambda$ is an eigenvalue of αA.
    (ii) $\lambda + c$ is an eigenvalue of $A + cI$.
12. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be distinct eigenvalues of $n \times n$ matrix A. Let $X_1, X_2, \cdots, X_k$ be the corresponding eigenvectors. Then show that $\{X_1, X_2, \cdots, X_k\}$ is a linearly independent set.