Con. 1451-13.

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KN–1815

(3 Hours) **[Total Marks : 100**

N.B.: (1) **All questions are compulsory.**

(2) Attempt any **two** subquestions from **part(a)**, **part(b)** and **part(c)**.

(3) Figures to the right indicate marks for respective subquestions.

1. (a) (i) Let $f : [0,2] \to \mathbb{R}$ be defined by

$$g(x) = x^2 \text{ if } 0 \le x < 1$$
$$= 2 \text{ if } x = 1$$
$$= 2x - 1 \text{ if } 1 < x \le 2$$

Let $P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ and $Q = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\}$. Find $L(P,f)$, $L(Q,f)$, $U(P,f)$ and $U(Q,f)$. Verify that $L(P,f) \le L(Q,f) \le U(Q,f) \le U(P,f)$. (5)

(ii) Discuss the pointwise and uniform convergence of the series of functions $\sum_{n=1}^{\infty} \frac{1}{(nx)^2}$, where $x \neq 0$ (5)

(b) (i) State the change of variable formula for triple integral, stating clearly the condition under which it is valid. Express how will you use it to express the triple integral in spherical co-ordinates (ρ, θ, φ). (5)

(ii) Show that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,0)$ where (5)

$$f(x,y) = \begin{cases} (x \cos \frac{1}{y}, y \cos \frac{1}{x}) & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

(c) (i) Evaluate the surface integral $\iint_S (x^2 z + y^2 z) dS$ where S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \ge 0$. (5)

(ii) Use the Divergence Theorem to calculate the surface integral $\iint_S \overline{F} \cdot \widehat{n} dS$, where $\overline{F}(x,y,z) = 3xy\widehat{i} + y^2\widehat{j} - x^2y^2\widehat{k}$, S is the surface of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. (5)

2. (a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Show that f is Riemann integrable on $[a,b]$ if and only if for each $\varepsilon > 0$, $\exists$ a partition P_ε of $[a,b]$ such that $U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$. (10)

(b) (i) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Show that f is Riemann integrable on $[a,b]$. (6)

(ii) Let $f : [0,3] \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$. Show that f is Riemann integrable. (4)

(c) (i) State and prove the Second Fundamental Theorem of Calculus. (6)

(ii) Evaluate the sum $\frac{1}{n+1} + \cdots + \frac{1}{2n}$ by expressing it as a Riemann sum of a suitable function. (as $n \to \infty$) (4)

3. (a) Let $\{f_n\}$ be a sequence of continuous real valued functions defined on a non-empty subset S of $\mathbb{R}$. If $\{f_n\}$ converges uniformly to a function f on S, then show that f is continuous on S. Further show that

$$\lim_{n \to \infty} \lim_{x \to p} f_n(x) = \lim_{x \to p} \lim_{n \to \infty} f_n(x)$$

for each $p \in S$. (10)

(b) (i) Let $\{f_n\}$ be a sequence of real values functions defined on a non-empty subset S of $\mathbb{R}$. Let $M_n = \sup\{|f_n(x) - f(x)| : x \in S\}$. Show that $\{f_n\}$ converges uniformly to f on S if and only if $\lim_{n \to \infty} M_n = 0$. (6)

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(ii) Let $f_n(x) = \frac{x^n}{n}$ for $x \in [0,1]$. Show that the sequence $\{f_n\}$ of differentiable function converges uniformly to a differentiable function f on $[0,1]$. Let g be the pointwise limit of $\{f_n'\}$ on $[0,1]$. Is $f'(1) = g(1)$? Does $\{f_n'\}$ converge uniformly to g on $[0,1]$? Justify your answer. (4)

(c) (i) State and prove Weierstrass M-test. (6)

(ii) By integrating the power series for $\frac{1}{1+x}$ term by term in $(-1,1)$, show that $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ for $x \in (-1,1)$. (4)

4. (a) Let S be an open subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}^m$. Suppose $f(x) = (f_1(x), \ldots, f_m(x))$. Show that f is continuous at $a \in S$ if and only if each f_k is continuous at a, $1 \le k \le m$. (10)

(b) (i) State and prove the Mean Value Theorem of scalar fields. (6)

(ii) Determine the second order Taylor formula for the function $f(x,y) = e^{x+y}$ at $(0,0)$. (4)

(c) (i) Let f be a scalar field defined on an open subset S of $\mathbb{R}^n$. If f is homogeneous of degree p on S and f is differentiable at $x \in S$, then show that $x \cdot \nabla f(x) = pf(x)$. Also prove that if $x \cdot \nabla f(x) = pf(x)\ \forall\, x \in S$, then f is homogeneous of degree p over S. (6)

(ii) Compute the matrices $Dg(1,1)$, $D(f(g(1,1)))$ and $D(f \circ g(1,1))$, where $f(u,v) = (uv, u^2+v^2)$, $g(x,y) = (x+y, x-y)$ (4)

5. (a) State and prove Stoke's Theorem for an oriented smooth, simple parameterized surface in $\mathbb{R}^3$ bounded by a simple, closed curve traversed counter clockwise assuming general form of Green's Theorem. (10)

(b) (i) If S and C satisfy hypothesis of Stoke's Theorem and f, g have continuous second order partial derivative, prove with usual notations (6)

(i) $\int_C (f\nabla g) \cdot d\overline{r} = \iint (\nabla f \times \nabla g)\widehat{n}\,dS$

(ii) $\int_C (f\nabla f) \cdot d\overline{r} = 0$

(iii) $\int_C (f\nabla g + g\nabla f) \cdot d\overline{r} = 0$

(ii) Use Stoke's Theorem to evaluate $\int_C \overline{F} \cdot d\overline{r}$ where $\overline{F}(x,y,z) = xz\widehat{i} + 2xy\widehat{j} + 3xy\widehat{k}$, C is the boundary of the part of the plane $3x + y + z = 3$ in the first octant. (4)

(c) (i) Let $S = \overline{r}(T)$ be a smooth parametric surface in uv plane. Define area of S. If S is represented by an equation $z = f(x,y)$ then show that areal of S is given by

$$\iint_T \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$$

where T is projection of S on xy-plane. (6)

(ii) Prove the following identities, assuming S and V satisfy the conditions of the Divergence Theorem and scalar fields f and g, components of $\overline{F}$ have continuous partial derivatives, $\widehat{n}$ is unit outward normal. (4)

(i) $|V| = \frac{1}{3}\iint_V \overline{r} \cdot \widehat{n}\,dS$ where $\overline{r} = x\widehat{i} + y\widehat{j} + z\widehat{k}$ and $|V|$ = volume of V.

(ii) $\iint_S \text{curl}\overline{F} \cdot \widehat{n}\,dS = 0$.