

- N.B.: (1) All questions are compulsory.
 (2) Figures to the right indicate marks for respective subquestions.

1. (a) Answer any ONE
- State and prove the Cayley Hamilton theorem. (8)
 - For real vector space V and a subspace W of V define quotient space V/W . Let V be a finite dimensional real vector space and W be a subspace of V . Show that $\dim V/W = \dim V - \dim W$. (8)
- (b) Answer any TWO
- Define orthogonal matrix. Prove that a $n \times n$ matrix A is orthogonal if and only if the column vectors of A form an orthonormal basis of \mathbb{R}^n . (6)
 - Let V be a finite dimensional inner product space over \mathbb{R} . If $f : V \rightarrow V$ is an isometry, then show that there exists unique $x_0 \in V$ and an orthogonal linear transformation $T : V \rightarrow V$ such that $f(x) = T(x) + x_0, \forall x \in V$. (6)
 - For a linear transformation $T : P_3(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ is defined by $T(f(x)) = f'(x)$ ($P_3(\mathbb{R})$ is a vector space of polynomials over \mathbb{R} with degree ≤ 3 and the zero polynomial). Find $\ker T$ and $\text{Im} T$. Verify the fundamental theorem of homomorphism of vector spaces in case of T . (6)
 - Show that a 2×2 orthogonal matrix A with $\det A = 1$ is a matrix of rotation. (6)
2. (a) Answer any ONE
- Show that every real symmetric matrix is orthogonally diagonalizable. (8)
 - Show that a quadratic form $Q[X]$ can be reduced to standard form $\sum_{i=1}^n \lambda_i y_i^2$ by orthogonal change of variable $X = PY, X = [x_1 \ x_2 \ \dots \ x_n]^t, Y = [y_1 \ y_2 \ \dots \ y_n]^t$. Further show that $Q[X]$ is positive definite if and only if $\lambda_i > 0, \forall i$. (8)
- (b) Answer any TWO
- Let $A_{n \times n}$ be a real symmetric matrix. Show that $AX \cdot Y = X \cdot AY$ for all $X, Y \in \mathbb{R}^n$. Hence or otherwise prove that eigen vectors corresponding to distinct eigen values of A are orthogonal. (6)
 - Show that the characteristic roots of a real symmetric matrix are real. (6)
 - If $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$, then show that $A^{100} + A^{20} + I_3$ is diagonalizable. (6)

[P.T.O.]

iv. Let $A_{n \times n}$ be a real diagonalizable matrix having eigen values 1 and -1. Show that $A = A^{-1}$. (6)

3. (a) Answer any ONE

i. Define a cyclic group. If G is a cyclic group of order n generated by a then show that a^m generates G if and only if $\gcd(m, n) = 1$. (8)

ii. State and prove Lagrange's theorem. (8)

(b) Answer any TWO

i. List all the elements of the group of symmetries of a rectangle (which may not be a square) as permutation on vertices. Find order of each element of this group. Is the group cyclic? (6)

ii. Define a group homomorphism. Show that $f : G \rightarrow G$ defined by $f(x) = x^2$ is a group homomorphism if and only if G is abelian. (6)

iii. Show that $G = \{A = \begin{pmatrix} a & a \\ a & a \end{pmatrix} / a \in \mathbb{R}, a \neq 0\}$ is a group under matrix multiplication. (6)

iv. Show that every group of prime order is cyclic. (6)

4. Answer any THREE

(a) Let V be a vector space and W be a subspace of V . Show that that, for any $x, y \in V$ $W + x = W + y$ or $W + x \cap W + y = \phi$. (5)

(b) Find the orthogonal transformation in \mathbb{R}^3 which represents reflection with respect to $x - y + z = 0$. (5)

(c) Determine whether matrix $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ is diagonalizable or not. (5)

(d) Let A and B both square matrices of order n such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal. Show that $AB = BA$. (5)

(e) Construct the composition table of $U(12)$. Find the inverse of each of its elements. (5)

(f) Show that $f : (\mathbb{R}, +) \rightarrow (\mathbb{R}^+, \cdot)$ defined by $f(x) = e^x$ is a group isomorphism. (5)
