Topology of metric space

Duration 2 $\frac{1}{2}$Hrs Marks: 75

N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any one : (8)

   i. (X, d) is a metric space. Define limit point of $F \subseteq X$. Also show that F is a closed set if and only if F contains all it's limit points.

   ii. Let (X, d) be a metric space and Y be a non-empty subset of X. Show that a subset G of Y is open in the subspace (Y, d) if and only if $G = V \cap Y$ where V is an open set in (X, d)

   (b) Attempt any Two : (12)

   i. Prove that $(\mathbb{N}, d)$ and $(\mathbb{N}, d_1)$ where d is the usual distance (induced from $\mathbb{R}$) and d_1 is the discrete metric in $\mathbb{N}$, are equivalent metric spaces.

   ii. State and prove Hausdorff property in a metric space (X, d)

   iii. Let (X, d) be a metric space and $A \subseteq X$. Let $\bar{A} = \{x \in X : \forall \epsilon > 0, B(x, \epsilon) \cap A \neq \emptyset\}$. Show that $\bar{A}$ is the smallest closed set containing A.

   iv. Find the interiors A^0, B^0 and the boundaries $\partial A, \partial B$ of $A, B \subset \mathbb{R}$ (distance being usual) where $A = (-2, 2) \cap \mathbb{Z}$, $B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$

2. (a) Attempt any one : (8)

   i. State and prove Cantor's Intersection Theorem for metric spaces.

   ii. (X, d) is a metric space. Show that a convergent sequence in (X, d) is a Cauchy sequence. Give an example to show that the converse is not true.

   (b) Attempt any Two : (12)

   i. Prove or disprove: Suppose d_1, d_2 are equivalent metrics on a non-empty set X. If (x_n) is bounded in (X, d_1) then (x_n) is bounded in (X, d_2).

   ii. (X, d) is a metric space and $A \subseteq X$. Prove that:
   (I) $X \setminus A^0 = \overline{X \setminus A}$
   (II) $X \setminus \bar{A} = (X \setminus A)^0$

   iii. Define a complete metric space and prove that a discrete metric space (X, d) is complete.

   iv. If (x_n) and (y_n) are sequences in a metric space (X, d) such that $(x_n) \longrightarrow p$ and $(y_n) \longrightarrow q$ then show that the sequence of real numbers $(d(x_n, y_n))$ converges to $d(p, q)$ in $(\mathbb{R}$, usual)

3. (a) Attempt any One : (8)

   i. Let $f : (X, d) \longrightarrow (Y, d')$ be a function. Prove that f is continuous on X if and only if for each open subset G of Y, $f^{-1}(G)$ is an open subset of X.

   ii. Let (X, d) and (Y, d') be metric spaces. Show that $f : X \longrightarrow Y$ is continuous on X if and only if for each subset A of X, $f(\bar{A}) \subseteq \overline{(f(A))}$

(b) Attempt any Two : (12)

i. Let (X, d) be a metric space and $f : X \longrightarrow \mathbb{R}$, ($\mathbb{R}$ with usual metric) is continuous on X. If $f(x_0) > 0$ for some $x_0 \in X$ then show that $\exists\ \delta > 0$ such that $f(x) > 0\ \forall x \in B(x_0, \delta)$.

ii. Show that the function $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[1, \infty]$ but not uniformly continuous on $(0, 1)$.

iii. Let (X, d) be a metric space and let $A \subseteq X$, If $d_A : X \longrightarrow \mathbb{R}$ is defined by $d_A(x) = d(x, A)$. Then show that d_A is continuous on X.

iv. Prove or disprove: If (X, d) and (Y, d') are metric spaces and $f : X \longrightarrow Y$ is a continuous bijective map, then for any open ball B in (X, d), $f(B)$ is an open ball in (Y, d').

4. Attempt any Three : (15)

(a) Show that $\| \ \|$ is a norm on X, where $X = M_2(\mathbb{R})$ and $\|A\| = \max_{1 \leq i,j \leq 2} \ \{|a_{ij}|\}$ for $A = (a_{ij})$

(b) Show that the set $G = \{(x_1, x_2) \in \mathbb{R}^2 : 2x_1 + 3x_2 < 1\}$ is an open set, distance in $\mathbb{R}^2$ being Euclidean.

(c) Let d and d_1 be equivalent metrics on X. If $(x_n) \longrightarrow p$ in (X, d) then prove that $(x_n) \longrightarrow p$ in (X, d_1)

(d) Let (X, d) be a metric space and $A \subseteq X$, then prove that A is dense if and only if for every nonempty open subset $G \subseteq X, G \cap A \neq \emptyset$.

(e) Show that $f : (\mathbb{R}^2, d_1) \longrightarrow (\mathbb{R}, d)$ defined by $f(x, y) = 2x + y$ is continuous on $\mathbb{R}^2$, where d_1 is Euclidean metric on $\mathbb{R}^2$ and d is usual metric on $\mathbb{R}$

(f) Let (X, d) and (Y, d') be metric spaces. Show that if $f : X \longrightarrow Y$ is uniformly continuous on X and if (x_n) in X is Cauchy then show that the sequence $(f(x_n))$ is Cauchy in Y.
