Inverse transformations
Generation of random observations from given distributions

Assume that random numbers, R_1, R_2, … are readily available, where each R_i itself is a random variable which is uniformly distributed over the range(0,1) . In other words Ri has pdf

$$f_R(x) = \begin{cases} 1, 0 \le x \le 1 \\ 0, otherwise \end{cases}$$

and cdf

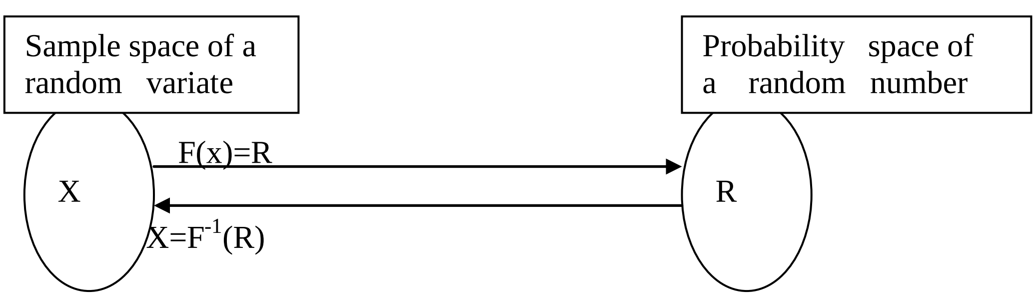
$$F_R(x) = \begin{cases} 0, x < 0 \\ x, 0 \le x \le 1 \\ 1, x > 1 \end{cases}$$

Throughout this chapter R and R_1, R_2, … represent random numbers uniformly distributed on (0, 1) and generated by one of the techniques or taken from a random-number table .

Inverse Transform Technique

The inverse transform technique can be used to sample from the uniform, the exponential, the Weibull, and the triangular distributions and empirical distributions. Additionally, it is the underlying principle for sampling from a wide variety of discrete distributions. The technique will be explained in detail for the exponential distribution and then applied to other distributions. It is the most straightforward, but not always the most efficient, technique computationally.

Let the p.d.f.of a random variate x be denoted f(x) and the c.d.f. be denoted F(x) . It can be shown that F(x) =R ~ U(0,1) ,*Since cdf lies between 0,1 it is u(o,1)*



1) *Uniform Distribution*

Consider a random variable X that is uniformly distributed on the interval [a, b]. A reasonable guess for generating X is given by

$$X = a + (b - a) R \qquad (1.1)$$

The pdf of X is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a}, a \le x \le b \\ 0, otherwise \end{cases}$$

The inverse transform technique can be utilized, at least in principle, for any distribution, but it is most useful when the cdf, $F(x)$,is of such simple form that its inverse, F^{-1}, can be easily computed. [1]A step-by-step procedure for the inverse transform technique,

The derivation of Equation (1.1) follows steps 1 through 3

Step 1. The cdf is given by

$$F(x) = \begin{cases} 0, x < a \\ \dfrac{x-a}{b-a}, a \le x \le b \\ 1, x > b \end{cases}$$

Step 2. Set $F(X) = (X - a) / (b - a) = R$.

Step 3. Solving for X in terms of R yields $X = a + (b - a) R$, which agrees with Equation (1.1).

Example Given a=2 and b= 4 generate six random observations using random numbers, 0.3, 0.48, 0.36, 0.01, 0.54, 0.34

Sr. no	Random number R	Random observation $X = a + (b - a) R$=2+2R
1	0.3	2.6
2	0.48	2.96
3	0.36	2.72
4	0.01	2.02
5	0.54	3.08
6	0.34	2.68

2) Exponential Distribution

The exponential distribution, has probability density function (pdf)

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, x \geq 0 \\ 0, x < 0 \end{cases}$$

and cumulative distribution function (cdf) given by

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 1 - e^{-\lambda x}, x \geq 0 \\ 0, x < 0 \end{cases}$$

The parameter λ can be interpreted as the mean number of occurrences per time unit. For example, if interarrival times $X_1, X_2, X_3, \ldots$ had an exponential distribution with rate λ, then λ could be interpreted as the mean number of arrivals per time unit, or the arrival rate. Notice that for any *i*

$$E(X_i) = \frac{1}{\lambda}$$

So that $1/\lambda$ is the mean interarrival time. The goal here is to develop a procedure for generating values $X_1, X_2, X_3, \ldots$ which have an exponential distribution.

Step 1. Compute the cdf of the desired random variable X. For the exponential distribution, the cdf is $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$.
Step 2. Set $F(X) = R$ on the range of X. For the exponential distribution, it becomes $1 - e^{-\lambda X} = R$ on the range $x \geq 0$. Since X is a random variable (with the exponential distribution in this case), it follows that $1 - e^{-\lambda X}$ is also a random variable, here called R. As will be shown later, R has a uniform distribution over the interval (0, 1).
Step 3. Solve the equation $F(X) = R$ for X in terms of R. For the exponential distribution, the solution proceeds as follows:

$$1 - e^{-\lambda X} = R$$

$$e^{-\lambda X} = 1 - R$$

$$-\lambda X = \ell n(1 - R)$$

$$X = -\frac{1}{\lambda}\ell n(1 - R) \qquad (2.1)$$

Equation (2.1) is called a random-variate generator for the exponential distribution. In general, Equation (2.1) is written as $X = F^{-1}(R)$. Generating a sequence of values is accomplished through step 4.

Step 4. Generate (as needed) uniform random numbers $R_1, R_2, R_3, \ldots$ and compute the desired random variates by

$$X_i = F^{-1}(R_i)$$

For the exponential case, $F^{-1}(R) = (-1/\lambda)\, \ell n\,(1 - R)$ by Equation (2.1), so that

$$X_i = -\frac{1}{\lambda}\ln(1 - R_i) \qquad (2.2)$$

for $i = 1, 2, 3, \ldots$. One simplification that is usually employed in Equation (2.2) is to replace $1 - R_i$ by R_i to yield

$$X_i = -\frac{1}{\lambda}\ln R_i \qquad (2.3)$$

Which is justified since both R_i and $1 - R_i$ are uniformly distributed on (0, 1).

Example : Generation of five Exponential Variates X_i with Mean 1,

$1/\lambda=1$ i.e. $\lambda=1$, $F(x) = 1 - e^{-x}$

$X_i = -\mathrm{Ln} R_i$

Given Random Numbers R_i

i	*1*	*2*	*3*	*4*	*5*
R_i	0.1306	0.0422	0.6597	0.7965	0.7696
X_i	0.1400	0.0431	1.078	1.592	1.468

3)Weibull Distribution

The Weibull distribution when the location parameter ν is set to 0, its pdf is given by Equation as

$$f(x) = \begin{cases} \dfrac{\beta}{\alpha^{\beta}} x^{\beta-1} e^{-(x/\alpha)^{\beta}}, & x \geq 0 \\ 0, \text{otherwise} \end{cases}$$

Where α > 0 and β > 0 are the scale and shape parameters of the distribution. To generate a Weibull variate, follow steps 1 through 3

Step 1. The cdf is given by

$$F(x) = 1 - e^{(\frac{x-\gamma}{\alpha})^{\beta}} \qquad x>\nu$$

$$F(x) = 1 - e^{(\frac{x}{\alpha})^{\beta}} \qquad x>0$$

Step 2. Let $F(x) = 1 - e^{(\frac{x}{\alpha})^{\beta}}$ = *R*.

Step 3. Solving for *X* in terms of *R* yields

$$X = \alpha[-\ell n\,(1- R)]^{1/\beta} \qquad (3.1)$$

it can be seen that if *X* is a Weibull variate, then X^{β} is an exponential variate with mean α^{β}. Conversely, if *Y* is an exponential variate with mean μ, then $Y^{1/\beta}$ is a Weibull variate with shape parameter β and scale parameter $\alpha = \mu^{1/\beta}$.

*4)**Triangular Distribution***

Consider a random variable *X* which has pdf

$$f(x) = \begin{cases} x & 0 < X < 1 \\ 2 - x, & 1 < X < 2 \\ 0 & otherwise \end{cases}$$

This distribution is called a triangular distribution with endpoints (0, 2) and mode at 1. Its cdf is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x^2}{2}, & 0 < X < 1 \\ 1 - \dfrac{(2-x)^2}{2}, & 1 < X < 2 \\ 1, & otherwise \end{cases}$$

For $0 \le X \le 1$, $$R = \frac{X^2}{2}$$

and for $1 \le X \le 2$, $$R = 1 - \frac{(2-X)^2}{2}$$

$0 \le X \le 1$ implies that $0 \le R \le \frac{1}{2}$, in which case $X = \sqrt{2R}$.

$1 \le X \le 2$ implies that $\frac{1}{2} \le R \le 1$, in which case $X = 2 - \sqrt{2(1-R)}$. Thus, X is generated by

$$X = \begin{cases} \sqrt{2R}, 0 \le R \le \dfrac{1}{2} \\ 2 - \sqrt{2(1-R)}, \dfrac{1}{2} < R \le 1 \end{cases}$$

Example Generate four random observations from triangular distribution over(0,1,2)

1. R=0.13 <0.5 X= $\sqrt{2R}$ =0.5099
2. R=0.04 <0.5 X= $\sqrt{2R}$ =0.2828
3. R= 0.65 >0.5 X= $2 - \sqrt{2(1-R)}$ =1.163
4. R=0.79 >0.5 X= $2 - \sqrt{2(1-R)}$ =1.3519

Normal distribution: consider random variable X which is normally distributed with mean μ and variance σ^2.

F(x) =P(X< x)=R

i.e. P(Z< (x-μ)/σ)=R

corresponding to R as area by using normal table we can read the value of the ordinate as z.

$z=\Phi^{-1}(R)$ $z=(x-\mu)/\sigma$ $x=\mu+\sigma z$

Example Service time of a bank teller is found to follow normal distribution with mean 5 and s.d. 1. Generate five service times.

i	*1*	*2*	*3*	*4*	*5*
R_i	0.1306	0.0422	0.6597	0.7965	0.7696
z_i	-1.1	-1.76	0.41	0.83	0.74
X_i	3.9	3.24	5.41	5.83	5.74

Empirical Continuous Distributions

If the modeler has been unable to find a theoretical distribution that provides a good model for the input data, then it may be necessary to use the empirical distribution of the data. One possibility is to simply resample the observed data itself. This is known as using the *empirical distribution*, and it makes particularly good sense when the input process is known to take on a finite number of values.

On the other hand, if the data are drawn from what is believed to be a continuous-valued input process, then it makes sense to interpolate between the observed data points to fill in the gaps. This section describes a method for defining and generating data from a continuous empirical distribution.

Example Five observations of fire crew response times (in minutes) to incoming alarms have been collected to be used in a simulation investigation possible alternative staffing and crew scheduling policies. The data are

2.76 1.83 0.80 1.45 1.24

Before collecting more data, it is desired to develop a preliminary simulation model which uses a response-time distribution based on these five observations. Thus, a method for generating random variates from the response-time distribution is needed. Initially, it will be assumed that response times X have a range $0 \le X \le c$, where c is unknown, but will be estimated by $\hat{c} = \max\{X_i : i = 1, \ldots, n\} = 2.76$, where $\{X_i, i = 1, \ldots, n\}$ are the raw data and $n = 5$ is the number of observations.

Table 8.2. Summary of Fire Crew Response-Time Data

i	Interval $x_{(i-1)} \le x \le x_{(i)}$	*Probability, 1/n*	*Cumulative Probability=i/n*	*Slope* $a_i = \dfrac{X(i)-X(i-1)}{1/n}$
1	$0.0 \le x \le 0.80$	0.2	0.2	4.00
2	$0.80 \le x \le 1.24$	0.2	0.4	2.20
3	$1.24 \le x \le 1.45$	0.2	0.6	1.05
4	$1.45 \le x \le 1.83$	0.2	0.8	1.90
5	$1.83 \le x \le 2.76$	0.2	1.0	4.65

Arrange the data from smallest to largest and let $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$ denote these sorted values. Since the smallest possible value is believed to be 0, define $x_{(0)} = 0$. Assign a probability of $1/n = 1/5$ to each interval $x_{(i-1)} \le x \le x_{(i)}$,
The slope of the i th line segment is given by

$$a_i = \frac{x_{(i)} - x_{(i-1)}}{1/n}$$

The inverse cdf is calculated by

$$X = \hat{F}^{-1}(R) = x_{(i-1)} + a_i\left(R - \frac{(i-1)}{n}\right)$$

when $(i - 1)/n < R \le i / n$.

For example, if a random number $R_1 = 0.71$ is generated, then R_1 is seen to lie in the fourth interval (between 3/5 = 0.60 and 4/5 = 0.80),,

$$\begin{aligned} X_1 &= x_{(4-1)} + a_4 (R_1 - (4 - 1) / n) \\ &= 1.45 + 1.90 (0.71 - 0.60) \\ &= 1.66 \end{aligned}$$

If a large sample of data is available (and sample sizes from several hundred to tens of thousands are possible with modern, automated data collection), then it may be more convenient and computationally efficient to first summarize the data into a frequency distribution with a much smaller number of intervals and then fit a continuous empirical cdf to the frequency distribution. Only a slight generalization of the above Equation is required to accomplish this. Now the slope of the ith line segment is given by

$$a_i = \frac{x_{(i)} - x_{(i-1)}}{c_i - c_{i-1}}$$

Where c_i is the cumulative probability of the first i intervals of the frequency distribution and $x_{(i-1)} \le x \le x_{(i)}$ is the ith interval. The inverse cdf is calculated by

$$X = \hat{F}^{-1}(R) = x_{(i-1)} + a_i(R - c_{i-1})$$

when $c_{i-1} < R \leq c_i$

Example

Suppose that 100 broken-widget repair times have been collected. The data are summarized in the following Table in terms of the number of observations in various intervals. For example, there were 31 observations between 0 and 0.5 hour, 10 between 0.5 and 1 hour, and so on.

Suppose it is known that all repairs take at least 15 minutes, so that $X \geq 0.25$ hour always. Then we set $x_{(0)} = 0.25$, as shown in Table .

Table Summary of Repair-Time Data

i	*Interval (Hours)*	*Frequency*	*Relative Frequency*	*Cumulative Frequency, c_i*	*Slope, a_i*
1	$0.25 \leq x < 0.5$	31	0.31	0.31	0.81
2	$0.5 \leq x < 1.0$	10	0.10	0.41	5.0
3	$1.0 \leq x < 1.5$	25	0.25	0.66	2.0
4	$1.5 \leq x \leq 2.0$	34	0.34	1.00	1.47

For example, suppose the first random number generated is $R_1 = 0.83$.
Then since R_1 is between $c_3 = 0.66$ and $c_4 = 1.00$, X_1 is

$$X_1 = x_{(4-1)} + a_4(R_1 - c_{4-1}) = 1.5 + 1.47\,(0.83 - 0.66) = 1.75$$

As another illustration, suppose that $R_2 = 0.33$. Since $c_1 = 0.31 < R_2 \leq 0.41 = c_2$,

$$\begin{aligned} X_2 &= x_{(1)} + a_2(R_2 - c_1) \\ &= 0.5 + 5.0\,(0.33 - 0.31) \\ &= 0.6 \end{aligned}$$

Discrete Distribution

All discrete distributions can be generated using the inverse transform technique, either numerically through a table-lookup procedure, or in some cases algebraically with the final generation scheme in terms of a formula. Other techniques are sometimes used for certain distributions, such as the convolution technique for the binomial distribution. Some of these methods are discussed in later sections. This subsection gives examples covering both empirical distributions and two of the standard discrete distributions, the (discrete) uniform and the geometric.

Example

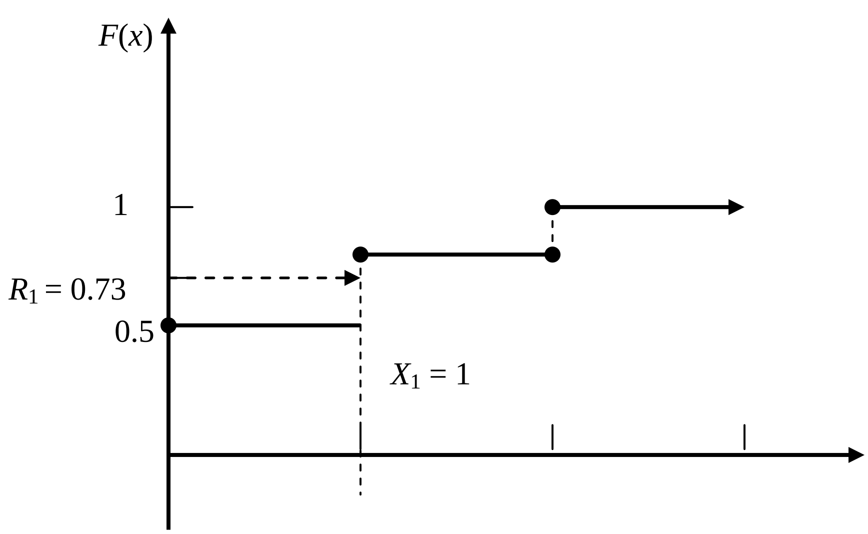
At the end of the day, the number of shipments on the loading dock of the IHW Company (whose main product is the famous, incredibly huge widget) is either 0, 1, or 2, with observed relative frequency of occurrence of 0.50, 0.30, and 0.20, respectively. Internal consultants have been asked to develop a model to improve the efficiency of the loading and hauling operations, and as part of this model they will need to be able to generate values, X, to represent the number of shipments on the loading dock at the end of each day. The consultants decide to model X as a discrete random variable with distribution as given below

The probability mass function (pmf), $p(x)$, is given by

$$p(0) = P(X = 0) = 0.50$$
$$p(1) = P(X = 1) = 0.30$$
$$p(2) = P(X = 2) = 0.20$$

and the cdf, $F(x) = P(X \leq x)$, is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.5, & 0 \leq x < 1 \\ 0.8, & 1 \leq x < 2 \\ 1.0, & 2 \leq x \end{cases}$$



0 1 2 3 *x*

The cdf of number of shipments, *X*.

Table Table for Generating the Discrete Variate *X*

i	*Input,* r_i	*Output,* x_i
1	0.50	0
2	0.80	1
3	1.00	2

Recall that the cdf of a discrete random variable always consists of horizontal line segments with jumps of size $p(x)$ at those points, x, which the random variable can assume.
$p(0) = 0.5$ at $x = 0$, of size $p(1) = 0.3$ at $x = 1$, and of size $p(2) = 0.2$ at $x = 2$.

Let $R_1 = 0.73$ Here $R_1 = 0.73$ is transformed to $X_1 = 1$.

In general, for $R = R_1$, if

$$F(x_{i-1}) = r_{i-1} < R \leq r_i = F(x_i)$$

Since $r_1 = 0.5 < R_1 = 0.73 \leq r_2 = 0.8$, set $X_1 = x_2 = 1$. The generation scheme is summarized as follows:

$$X = \begin{cases} 0, & 0 < R < 0.5 \\ 1, & 0.5 \leq R < 0.8 \\ 2, & 0.8 \leq R < 1 \end{cases}$$

Example (A Discrete Uniform Distribution)

Consider the discrete uniform distribution on $\{1, 2, \ldots, k\}$ with pmf and cdf given by

$$p(x) = \frac{1}{k}, x = 1, 2, \ldots, k$$

and

$$F(x)=\begin{cases}0, x<1\\ \frac{1}{k}, 1\le x<2\\ \frac{2}{k}, 2\le x<3\\ \vdots\\ \frac{k-1}{k}, k-1\le x<k\\ 1, k\le x\end{cases}$$

Let $x_i = i$ and $r_i = p(1) + \ldots + p(x_i) = F(x_i) = i / k$ for $i = 1, 2, \ldots, k$. Then by using Inequality (8.13) it can be seen that if the generated random number R satisfies

$$r_{i-1} = \frac{i-1}{k} < R \le r_i = \frac{i}{k} \qquad \text{(A)}$$

Then X is generated by setting $X = i$. Now, Inequality (A) can be solved for i:

$$i - 1 < Rk \le i$$

$$Rk \le i < Rk + 1$$

Let $[y]$ denote the smallest integer $\ge y$. for example, $[7.82] = 8$, $[5.13] = 6$, and $[-1.32] = -1$. For $y \ge 0$, $[y]$ is a function that rounds up. This notation and Inequality yield a formula for generating X, namely

$$X = [Rk] \qquad \text{(B)}$$

For example, consider generating a random variate X, uniformly distributed on $\{1, 2, \ldots, 10\}$. The variate, X, might represent the number of pallets to be loaded onto a truck. Using Table A.1 as a source of random numbers, R, and Equation (B) with $k = 10$ yields

$R_1 = 0.78,$ $X_1 = [7.8] = 8$
$R_2 = 0.03,$ $X_2 = [0.3] = 1$
$R_3 = 0.23,$ $X_3 = [2.3] = 3$
$R_4 = 0.97,$ $X_4 = [9.7] = 10$

The procedure discussed here can be modified to generate a discrete uniform random variate with any range consisting of consecutive integers

Example

Consider the discrete distribution with pmf given by

$$p(x) = \frac{2x}{k(k+1)}, x = 1,2,..., k$$

For integer values of x in the range {1, 2, …, k}, the cdf is given by

$$F(x) = \sum_{i=1}^{x} \frac{2i}{k(k+1)}$$

$$= \frac{2}{k(k+1)} \sum_{i=1}^{x} i$$

$$= \frac{2}{k(k+1)} \frac{x(x+1)}{2}$$

$$= \frac{x(x+1)}{k(k+1)}$$

Generate R and use Inequality to conclude that $X = x$ whenever

$$F(x-1) = \frac{(x-1)x}{k(k+1)} \le R < \frac{x(x+1)}{k(k+1)} = F(x)$$

$$(x - 1)\, x \le k\,(k + 1)\, R < x\,(x + 1)$$

To solve this inequality for x in terms of R, first find a value of x that satisfies

$$(x - 1)\, x = k\,(k + 1)\, R$$

or

$$x^2 - x - k\,(k + 1)\, R = 0$$

Then by rounding up, the solution is $X = [x\text{-}1]$. By the quadratic formula, namely

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 1$, $b = -1$, $c = -k\,(k + 1)\,R$, the solution to the quadratic equation is

$$x = \frac{1 \pm \sqrt{1 + 4k(k+1)R}}{2}$$

The positive root of this Equation is the correct one to use so X is generated by

$$X = \left[\frac{1 + \sqrt{1 + 4k(k+1)R}}{2} - 1 \right]$$

Example (The Geometric Distribution)

Consider the geometric distribution with pmf

$$p(x) = p\,(1 - p)^x, \qquad x = 0, 1, 2, \ldots$$

where $0 < p < 1$. Its cdf is given by

$$F(x) = \sum_{j=0}^{x} pq^j$$

$$= \frac{p\left(1-q^{x+1}\right)}{1-q}$$

$$= 1-q^{x+1}$$

For x = 0, 1, 2,… Using the inverse transform technique geometric random variable X will assume the value x whenever

$$F(x-1) = 1-(1-p)^x \leq R < 1-(1-p)^{x+1} = F(x)$$

where R is a generated random number assumed $0 < R < 1$. Solving Inequality (8.19) for x proceeds as follows:

$$(1-p)^{x+1} \leq 1-R < (1-p)^x$$
$$(x+1)\ell n(1-p) \leq \ell n(1-R) < x\ell n(1-p)$$

But $1- p < 1$ implies that $\ell n\,(1 - p) < 0$, so that

$$\frac{\ell n(1-R)}{\ell n(1-p)} - 1 \leq x < \frac{\ell n(1-R)}{\ell n(1-p)}$$

Thus, $X = x$ for that integer value of x satisfying Inequality or, in brief, using the round-up function [.]

$$X = \left[\frac{\ell n(1-R)}{\ell n(1-p)} - 1\right] \qquad \text{(A)}$$

Since p is a fixed parameter, let $\beta = -1/\ell n\,(1 - p)$. Then $\beta > 0$ and, by Equation (A), $X = [-\beta\ell n(1 - R) - 1]$.

Occasionally, a geometric variate X is needed which can assume values {a, a + 1, a + 2, …} with pmf $p(x) = p(1 - p)^{x-a}$ (x = a, a + 1,….). Such a variate, X can be generated, using Equation (A), by

$$X = a + \left[\frac{\ell n(1+R)}{\ell n(1-p)} - 1\right] \qquad \text{(B)}$$

Example

Generate three values from a geometric distribution on the range $\{X \geq 1\}$ with mean 2. Such a geometric distribution has pmf $p(x) = p(1 - p)^{x-1}$ (x = 1, 2,….)with mean $1/p = 2$, or $p = 1/2$. Thus, X can be generated by Equation (B) with a = 1, $p = ½$, and $1/\ell n(1 - p) = -1.443$. Using random number table A.1, $R_1 = 0.932$, $R_2 = 0.105$, and $R_3 = 0.687$, which yields

$$X_1 = 1 + [-1.443\ \ell n(1 - 0.932) - 1]$$
$$= 1 + [3.878 - 1] = 4$$
$$X_2 = 1 + [-1.443\ \ell n(1 - 0.105) - 1] = 1$$
$$X_3 = 1 + [-1.443\ \ell n(1 - 0.687) - 1] = 2$$

Convolution method

The probability distribution of a sum of two or more independent random variables is called a *convolution* of the distributions of the original variables.

- Erlang distribution.
  - An Erlang random variable X with parameters (K,Ө) can be shown to be the sum of K independent exponential random variables X_i ,i= 1,2,3…..K each having a mean 1/Ө

$$X = \sum_{i=1}^{K} X_i$$

  - Using equation that can generate exponential variable, an Erlang variate can be generated by

$$X = \sum_{i=1}^{K} \frac{-1}{\theta} lnRi = \frac{-1}{\theta}\ ln \prod_{i=1}^{k} Ri$$

Accept Reject technique

Example: Steps to generate uniformly distributed random numbers between 1/4 and 1.

Step 1. Generate a random number *R*
Step 2a. If, R≥ ¼ accept *X* = *R*, go to Step 3
Step 2b. If, R<1/4 reject *R*, return to Step 1
Step 3. If another uniform random variate on [1/4, 1] is needed, repeat the procedure beginning at Step 1. Otherwise stop.

- **Poisson Distribution** : The Pmf is

$$P(X) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x=0,1,2\ldots\ldots\ldots.$$

where X can be interpreted as the number of arrivals in one unit time.

  - From the original Poisson process definition, we know the inter arrival time,t_1, t_2,t_3…..are exponentially distributed with a mean of λ, i.e. λ arrivals in one unit time.
  - Relation between the two distribution:

$$X=n$$

if and only if

$$A_1 + A_2 + \ldots + A_n \leq 1 < A_1 + A_2 + \ldots + A_n + A_{n+1}$$

essentially this means if there are *n* arrivals in one unit time, the sum of interarrival time of the past *n* observations has to be less than or equal to one, but if one more interarrival time is added, it is greater then one (unit time).

  - The ti s in the relation can be generated from uniformly distributed random number ti, thus

$$\sum_{i=1}^{n} \frac{-1}{\lambda}\, lnRi \leq 1 < \sum_{i=1}^{n+1} \frac{-1}{\lambda}\, lnRi$$

that is

$$\prod_{i=1}^{n} Ri \geq e^{-\lambda} > \prod_{i=1}^{n+1} Ri$$

  - Now we can use the Acceptance-Reject method to generate Poisson distribution.

Step 1. Set $n = 0, P = 1$.
Step 2. Generate a random number R_{n+1} and replace P by $P*R_{n+1}$.
Step 3. If $P < e^{-\lambda}$, then accept $N = n$, meaning at this time unit, there are n arrivals. Otherwise, reject the current n, increase n by one, return to Step 2.

Exercises

1. Develop a random-variate generator for a random variable X with the pdf

$$f(x) = \begin{cases} e^{2x}, -\infty < x \le 0 \\ e^{-2x}, 0 < x < \infty \end{cases}$$

2. Develop a generation scheme for the triangular distribution with pdf

$$f(x) = \begin{cases} \frac{1}{2}(x-2), 2 \le x \le 3 \\ \frac{1}{2}(2-\frac{x}{3}), 3 < x \le 6 \\ 0, otherwise \end{cases}$$

Generate 10 values of the random variate, compute the sample mean, and compare it to true mean of the distribution.

3. Given the following cdf for a continuous variable with range -3 to 4, develop a generator for the variable.

$$F(x) = \begin{cases} 0, x \le -3 \\ \frac{1}{2} + \frac{x}{6}, -3 < x \le 0 \\ \frac{1}{2} + \frac{x^2}{32}, 0 < x \le 4 \\ 1, x > 4 \end{cases}$$

4. Given the pdf $f(x) = x^2/9$ on $0 \le x \le 3$, develop a generator for this distribution.