Con. 1715-13. **CG–2309**

(2½ Hours) **[Total Marks : 60**

N.B.: (1) **All questions are compulsory.**
(2) Figures to the right indicate marks for respective subquestions.

1. (a) State and prove the First Isomorphism Theorem of vector spaces. (7)
   (b) Attempt any **two** questions:
      (i) Let V be an n-dimensional inner product space over $\mathbb{R}$ and W be a subspace of V such that $\dim W = n-1$. Let u be a unit vector orthogonal to W. Show that $T: V \to V$ defined by $Tv = v - 2\langle v, u\rangle u$ is an orthogonal linear transformation such that $T(w) = w, \ \forall w \in W$ and $T(u) = -u$. (4)
      (ii) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 5 & -3 & 0 \\ 2 & -3 & 0 \end{pmatrix}$. Using Cayley Hamilton Theorem, find $A^4 + A^3 - 5A^2 + A + 2I$. (4)
      (iii) Let A be a 3×3 real orthogonal matrix with $\det(A) = 1$, show that 1 is an eigenvalue of A. (4)
      (iv) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $\langle v, w\rangle = 0$ implies $\langle Tv, Tw\rangle = 0$ for all $v, w \in \mathbb{R}^2$. Show that $T = cS$ where $S: \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal transformation and c is a real constant. (4)

2. (a) Show that a real symmetric matrix A is orthogonally diagonalizable. (7)
   (b) Attempt any **two** questions:
      (i) If A is an $n \times n$ real matrix having n distinct eigenvalues, show that A is diagonalizable. Give an example to show that the converse of the above statement is not true. (4)
      (ii) Let A be a 3×3 non-zero nilpotent matrix over $\mathbb{R}$. Is A diagonalizable? Justify your answer. (4)
      (iii) Let A, B be $n \times n$ matrices over $\mathbb{R}$ such that $AB = A - B$. If B is a diagonalizable matrix with only one eigenvalue 2, then show that A is diagonalizable. (4)
      (iv) If A, B are $n \times n$ positive definite matrices, prove that $A + B$ is positive definite. Is the converse of the above statement true? Justify your answer. (4)

3. (a) State and prove Lagrange's Theorem. (7)
   (b) Attempt any **two** questions:
      (i) Let G be a cyclic group of order n generated by $'a'$. Show that a^m generates G if and only if $(m, n) = 1$. (4)
      (ii) Let $f, g : G \to G'$ be group homomorphisms and $H = \{x \in G : f(x) = g(x)\}$. Prove or disprove: H is a subgroup of G. (4)
      (iii) Give an example of a group G and $a, b \in G$ such that $o(a) = o(b) = 2$ and $o(ab) = 5$. (4)
      (iv) Let G be a group of order n where n is not divisible by 3. If $(ab)^3 = a^3b^3$ for each $a, b \in G$, show that G is an abelian group. (4)

4. Attempt any **three** questions:
   (a) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$. A linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(X) = AX$ (X being a column vector in $\mathbb{R}^3$). Find $\ker T$, a basis of $\ker T$ and $\mathbb{R}^3/\ker T$. Also find Im T. (5)

[TURN OVER

(b) Let V be a finite dimensional inner product space over $\mathbb{R}$ and $g : V \to V$ be an isometry. Prove that there exists unique $x_0 \in V$ and an orthogonal transformation $T : V \to V$ such that $g(x) = T(x) + x_0$ for each $x \in V$. (5)

(c) Let $u, v \in \mathbb{R}^n$ be non-zero orthogonal column vectors and $A = uv^t$. Find eigenvalues of A. Is A diagonalizable? Justify your answer. (5)

(d) Show that the map $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ defined by $f(A) = (A^t)^{-1}$ is a group homomorphism. Is it an automorphism? Justify your answer. (5)